THE FEBRUARY MEETING IN NEW YORK

The three hundred eighty-fifth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 28, 1942. The attendance was approximately two hundred including the following one hundred forty-seven members of the Society:

R. L. Anderson, Lawrence Annenberg, R. G. Archibald, B. H. Arnold, K. J. Arnold, L. A. Aroian, Fred Assadourian, F. E. Baker, J. D. Bankier, Alfred Basch, P. T. Bateman, Stefan Bergman, Felix Bernstein, D. H. Blackwell, Gertrude Blanch, Samuel Borofsky, C. B. Boyer, A. D. Bradley, A. B. Brown, G. E. Bulloch, J. H. Bushey, J. W. Calkin, J. A. Clarkson, Harvey Cohn, K. S. Cole, L. M. Comer, T. F. Cope, Richard Courant, J. H. Curtiss, M. D. Darkow, Norman Davids, J. L. Doob, Jesse Douglas, Arnold Dresden, Nelson Dunford, Jacques Dutka, J. E. Eaton, M. L. Elveback, Paul Erdös, Aaron Fialkow, D. A. Flanders, A. I. Forsythe, Tomlinson Fort, R. M. Foster, G. A. Foyle, Hans Fried, K. O. Friedrichs, H. P. Geiringer, A. M. Gelbart, B. P. Gill, M. C. Gray, H. J. Greenberg, Bernard Greenspan, Lewis Greenwald, H. M. Griffin, E. J. Gumbel, Jacques Hadamard, D. W. Hall, F. C. Hall, N. A. Hall, M. H. Heins, Edward Helly, E. H. C. Hildebrandt, Banesh Hoffmann, T. R. Hollcroft, E. A. Hoy, E. M. Hull, W. A. Hurwitz, W. H. Ingram, G. K. Kalisch, Edward Kasner, L. S. Kennison, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, J. S. LeCaine, Solomon Lefschetz, Howard Levi, Madeline Levin, Norman Levinson, E. R. Lorch, A. N. Lowan, L. A. MacColl, C. C. MacDuffee, H. F. MacNeish, P. T. Maker, W. T. Martin, Imanuel Marx, Walther Mayer, H. L. Mintzer, E. C. Molina, Deane Montgomery, M. E. Munroe, F. J. Murray, C. A. Nelson, K. L. Nielsen, L. F. Ollmann, Oystein Ore, J. C. Oxtoby, A. S. Peters, R. S. Phillips, Harry Pollard, E. L. Post, Walter Prenowitz, M. H. Protter, H. A. Rademacher, M. S Rees, C. F. Rehberg, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Raphaël Salem, Arthur Sard, I. E. Segal, Seymour Sherman, Max Shiffman, L. L. Silverman, James Singer, M. F. Smiley, A. H. Sprague, Wolfgang Sternberg, J. J. Stoker, A. H. Stone, R. C. Strodt, W. C. Strodt, Alvin Sugar, M. M. Sullivan, J. D Tamarkin, J. W. Tukey, J. L. Vanderslice Wolfgang Wasow, Louis Weisner, M. E. Wells, W. F. Whitmore, Nobert Wiener, John Williamson, Audrey Wishard, H. A. Wood, M. A. Woodbury, Leo Zippin.

Professor J. H. Curtiss presided at the morning session which was devoted to short papers. At the afternoon session, at which Vice President J. D. Tamarkin presided, an address entitled *On a general equation for relaxation oscillations* was given by Professor Norman Levinson.

Titles and cross references to the abstracts of papers read at this meeting follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–10 were read before the morning session and papers numbered 11–27 were read by title. Paper 6 was read by Professor Silverman and paper 7 by

Professor Kasner. Dr. Mann was introduced by Professor H. A. Rademacher.

1. Wolfgang Wasow: *On boundary layer problems in the theory of ordinary differential equations.* (Abstract 48-3-121.)

2. Harry Pollard: *The generalized Stieltjes transform.* (Abstract 48-3-117.)

3. K. L. Nielson: *Some properties of functions satisfying partial differential equations of elliptic type.* (Abstract 48-3-116.)

4. A. M. Gelbart: *Functions of two variables with bounded real parts in domains not equivalent to the bicylinder.* (Abstract 48-3-110.)

5. Raphaël Salem: *On singular monotonic functions of the Cantor type.* (Abstract 48-3-118.)

6. L. L. Silverman and Otto Szász: *On a class of Nörlund matrices.* (Abstract 48-3-120.)

7. Edward Kasner and J. J. DeCicco: *A generalization of the isothermal transformations.* (Abstract 48-3-131.)

8. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* Preliminary report. (Abstract 47-11-469.)

9. Stefan Bergman: *Two-dimensional flow around two profiles.* (Abstract 48-3-122.)

10. H. B. Mann: *Proof of the conjecture on the density of sums of sets of positive integers.* (Abstract 48-3-105.)

11. A. A. Albert: *Non-associative algebras.* II. *New simple algebras.* (Abstract 48-3-102-*t*.)

12. P. O. Bell: *The parametric osculating quadrics of a family of curves on a surface.* (Abstract 48-3-125-*t*.)

13. Paul Civin: *Two-to-one mappings of three-dimensional sets.* (Abstract 48-3-137-*t*.)

14. Nathaniel Coburn: *Conformal geometry of unitary space.* (Abstract 48-3-126-*t*.)

15. Nathaniel Coburn: *Congruences in unitary space.* (Abstract 48-3-127-*t*.)

16. J. J. DeCicco: *A generalization of the dual-isothermal transformations.* (Abstract 48-3-128-*t*.)

17. J. J. DeCicco: *Point transformations by which straight lines correspond to circles.* (Abstract 48-3-129-*t*.)

18. Jesse Douglas: *On the geodesic surfaces of a given curve family.* (Abstract 48-3-130-*t*.)

19. H. L. Garabedian: *Hausdorff integral transformation.* (Abstract 48-3-109-*t*.)

20. H. J. Greenberg and H. S. Wall: *Hausdorff means included between* $(C, 0)$ *and* $(C, 1)$. (Abstract 48-3-111-*t*.)

21. Irving Kaplansky: *Note on a common error concerning kurtosis.* (Abstract 48-3-136-*t*.)

22. Edward Kasner and Don Mittleman: *A general theorem on the initial curvatures of dynamical trajectories.* (Abstract 48-3-132-*t*.)

23. A. N. Lowan, Gertrude Blanch, and William Horenstein: *Inversion of the q-series associated with Jacobi elliptic functions.* (Abstract 48-3-114-*t*.)

24. A. N. Lowan and Abraham Hillman: *A short table of the zeros of the equation* $f(x)=J_0(x)Y_0(kx)-J_0(kx)Y_0(x)=0$. (Abstract 48-3-115-*t*.)

25. A. R. Schweitzer: *Note on functions which generate an abstract field.* (Abstract 48-3-106-*t*.)

26. Alexander Weinstein: *Spherical pendulum and complex integration.* (Abstract 48-3-124-*t*.)

27. Don Mittleman: *Spin in newtonian mechanics.* (Abstract 48-5-210-*t*.)

T. R. HOLLCROFT,
Associate Secretary

Research—A National Resource. II. *Industrial Research.* (Report of the National Research Council to the National Resources Planning Board.) Washington, D.C., United States Government Printing Office, 1941. 11+370 pp. $1.00.

This report, based on a canvass of industrial laboratories, gives an account of the nature and extent of present day industrial research in this country. It consists of an organized collection of over twenty studies, prepared by specially qualified men, covering such topics as the development of industrial research, fundamental research in industry, careers in research, research in the aeronautics, petroleum and steel industries, location and extent of research, and research abroad. There are studies on the role of chemistry, of physics, biology, mathematics, and various fields of engineering, in industrial research. The part on mathematics (pp. 268–288), prepared by Dr. T. C. Fry, will be found interesting and valuable to all mathematicians.

Dr. Fry's report, entitled *Industrial mathematics*, has been reprinted as a supplementary issue of the American Mathematical Monthly (vol. 48 (1941), no. 6, part 2), and also in the Bell System Technical Journal (vol. 20 (1941), pp. 255–292). It begins with a discussion of the nature of the industrial mathematician's work, followed by a list of the qualifications that these men should possess, pointing out that the mathematician in industry must generally function as a consultant rather than as a project man. The lack of a center of training for such men is stressed. In recent months steps have been taken to improve this situation. A study of the uses of mathematics in industry and a section on statistics make up the rest of the report. The many interesting examples give a vivid impression of the growing use of mathematics in industry.

R. V. CHURCHILL

The Analysis of Economic Time Series, by Harold T. Davis. (Cowles Commission for Research in Economics, Monograph no. 6.) Bloomington, Indiana, Principia Press, 1941. 14+620 pp. $5.00.

It is unfortunate and a sign of the extreme specialization of science and the scientists that many advances made in related fields are ignored or neglected by the mathematicians. This is not altogether their fault, because nobody could possibly follow the ever growing literature in all phases of science utilizing mathematics. The whole mathematical profession and especially its statistical branch is in-

debted to Professor H. T. Davis, who has in this monumental volume assembled and lucidly interpreted a great number of the recent developments in the interesting, if somewhat puzzling, field of the study and analysis of economic time series. It also contains many of his own very important contributions. I feel certain that this book is destined to exert a deep influence upon the future development of the field. It also will stimulate the thinking of many who have been working on entirely different problems and who may find some of their difficulties related to the questions discussed in Professor Davis' book.

The first chapter deals with the history of the analysis of time series and gives a very valuable account of the most important writings in the field. It also affords an excellent introduction to the other parts of the book. The mathematical reader may not feel he can even glance through the whole 600 pages of the book. But he will be able to gain a rapid survey of the field and to select the material most valuable to himself by reading carefully the 60 pages of the first chapter.

The second chapter deals with the technique of harmonic analysis; this is the representation of empirical data by Fourier series. It does not on the whole add very much to the classical treatment of the problems. The last section relates the technique of harmonic analysis and more generally the method of decomposing empirical series by orthogonal function to the theory of multiple correlation. It contains some very valuable and original ideas. There are also, as in all other chapters, applications to economic data, which will be of great interest to economists.

The third chapter treats serial correlation. This is the phenomenon of the appearance of leads and lags among correlated variables. A special case is auto-correlation, where the individual items of one time series are correlated with each other. This chapter presents the best exposition of these extremely interesting and difficult problems which have given so much headache to the statistician. A great deal of the work originally done by Norbert Wiener is reproduced but important theorems are added. There are also new calculation techniques. The problem of continuous spectra is treated as well as some questions connected with random variation which have received so much attention lately. Finally, this chapter gives an application of the ideas of lag correlation to supply and demand curves, which leads to a discussion of the so-called cobweb theorem well known in economic statistics.

The fourth chapter brings an extensive discussion of the important problem of the theory of random series, which naturally is of great

interest not only to professional statisticians but also to workers outside the field of economic statistics. It has a definite relationship to some of the fundamental assumptions of probability theory, for instance, to the frequency definition of probability by von Mises. Goutereau's method is discussed and Yule's theory of random series is generalized with the help of the auto-correlation function. The theory of accumulated random series which has received some attention from mathematicians recently is presented as well as the influence of moving averages upon random series. The theory of sequences and reversals originally introduced by Bortkiewicz and recently further developed by H. E. Jones receives considerable attention. Finally, an application to the economic problems of stock market action is given.

Chapter 5 deals with the difficult and puzzling problem of degrees of freedom in economic time series, which has troubled statisticians for a long time. It is well known to the economic statistician that additional observations within the same time interval do not in general add materially to our information about economic phenomena we are studying with the help of statistical methods. If, however, the Fisherian theory of degrees of freedom is uncritically applied, it would seem that we are gaining a great deal of information. Professor Davis attacks the question as a problem in inverse probability essentially on the basis of Bayes' theorem. He considers particularly Schuster's, Walker's and Fisher's tests of significance in harmonic analysis and some problems recently discussed by Thurston, Frisch, and Hotelling in the field of factor analysis. Dissatisfied with previous results he finally proposes a very ingenious method which he calls the method of elementary energies.

Chapter 6 deals with the analysis of trends. The secular trend represents the non-periodic movements in economic time series. The discussion adds some new formulae for the correlation of residuals from polynomial trends. It treats the elimination of seasonal variation by the method of link relatives and the variate difference method which utilizes series of successive finite differences of the data. A very interesting section deals with the fitting of the logistic, which is a somewhat difficult problem because of the nature of the curve. Its solution is, however, important for social and economic statistics because of the role of the logistic in population theory and related fields. Hotelling's method of fitting the curve to its differential equation is discussed extensively as well as other approaches. Additional sections deal with the growth of population and of production. Finally Mr.

Leavens' interesting theory of frequency distributions of time series is presented.

Chapter 7 deals with periodogram analysis. This is one particular technique for the search for hidden periodicities. It utilizes the amplitudes of sine-cosine curves fitted to trial products. Davis presents some material on sine-cosine series, random series, smoothed and cumulated random series, and economic applications to various empirical data. A section deals with sunspot numbers and another with galvanometer series. It is a clear and valuable presentation of the technique of periodogram analysis.

Chapter 8 deals with the evidence and explanation of economic cycles. This is the much discussed and still largely unsolved problem of the somewhat irregular appearance of booms and depressions. Professor Davis first discusses and classifies economic business cycle theories and treats in more detail the mathematical theories of Evans, Roos, Frisch, Tinbergen, and Kalecki. He also goes into some problems on statistical hysteresis which are very interesting from a purely mathematical point of view. He discusses the methods proposed by Lotka, Volterra, Jones, and others in this field. The remainder of the chapter is devoted to some original work in the theory of crises which Professor Davis tries to explain by a sort of resonance theory. Chapter 9 deals with the nature of wealth and income and presents first statistical data on the personal distribution of income as well as the classical theory of Pareto's law. Professor Davis also tries to give a new and very interesting general distribution function which seems to fit the empirical income distribution better than Pareto's function or variants proposed by other writers. The chapter deals, however, mainly with economic and social problems.

Chapter 10 is entitled, "The dynamic trends from the point of view of the equation of exchange," and will not be of very great interest to non-economists.

Chapter 11 takes up very courageously the difficult problem of forecasting economic time series, especially as applied to stock prices. It reviews the classical work of Henry Schultz, *The standard error of a forecast from a curve.* Then it presents Professor Davis' own contribution which essentially assumes that it is not possible to forecast in the future beyond a range equivalent to the range of the available data. There are some interesting ideas about moving periodogram analysis and the method of probable error bands.

The last chapter finally deals with interpretation and critique of the material presented, especially from the point of view of business

cycles. It also gives a new "economic" interpretation of history and presents some interesting analogies between mathematics and economics.

This book should prove very useful to the statistician as a handbook. He will find a great deal of stimulation and valuable material and procedures even if he is not professionally concerned with economic problems. The pure mathematician can gain a survey of the field. He may be interested especially in some of the unsolved problems of serial correlation and periodogram analysis which evidently require very powerful tools of analysis. One has to be extremely grateful to the author for having accumulated so much material and presented it in such a readable and interesting form.

GERHARD TINTNER

A Survey of Modern Algebra. By Garrett Birkhoff and Saunders MacLane. New York, Macmillan 1941. 11+450 pp. $3.75.

In this book Professors Birkhoff and MacLane have made an important contribution to the pedagogy of algebra. Their emphasis is on the methods and spirit of modern algebra rather than on the subject matter for itself. The word "survey" in the title is quite accurate; for, although many topics are treated, none of them is really completely developed. The most important parts of each theory are included and that is all that can be asked of an introductory textbook.

Because of the authors' emphasis on "method" rather than "fact" the book will not be of much use as a reference work. But there is no dearth of good reference works in algebra, and in the reviewer's opinion the present textbook will prove more useful than another encyclopedic treatise would have been.

A discussion of the topics included will help to indicate the authors' purposes.

The first three chapters of the book are ostensibly devoted to the development of number systems: starting with postulates for the integers, then defining rational numbers in terms of the integers, and next giving an outline of the Dedekind construction of real numbers. Actually much more is happening. Such fundamental concepts as congruence, residue class, isomorphism, and ordered and well ordered sets are introduced and applied in a natural manner to the theory. Also the generalizations from integers to integral domain, and from rational number to field are made at suitable stages of the development. This procedure of starting with properties of a familiar mathematical system and generalizing to an abstract system is typical of

the spirit of the book and is used in the development of almost every topic treated.

The traditional theory of equations, which is all too frequently the undergraduate's only contact with algebra, places much emphasis on the solution of polynomial (and sometimes also non-algebraic) equations by approximation methods. This involves considerable attention to the analysis of polynomial functions, including such topics as Descartes' rules of signs, and Sturm's and Budan's theorems for isolation of roots. Such theorems may indeed be nice applications of the calculus but they certainly do not give the student any new general concepts either in algebra or analysis. A result of this procedure is that most beginning graduate students have not even the ghost of an idea as to what algebra is really about, even those parts of algebra that have proved to be of almost universal mathematical interest. (Contrast this with the situation in analysis where the undergraduate is getting acquainted with such basic ideas as continuity, limit, and infinite processes.)

In contrast to the traditional approach, in the authors' treatment (Chapters IV and V) the parts of the theory of polynomials and theory of equations that belong to analysis are omitted. (One major exception to this general policy is the inclusion of several sections on the fundamental theorem of algebra, but these can be skipped without loss of continuity.) Because of these omissions, an undergraduate course in algebra based on this book has time to include numerous more generally useful algebraic topics and methods; that is, the student can be presented with ideas that lead somewhere, rather than being sent up a blind alley.

Chapter VI deals with group theory. Only the most elementary (and therefore the most frequently needed) parts of the theory of abstract groups are included. The importance of groups of transformations and of permutation groups is emphasized throughout. As usual the topics treated are pegs upon which are hung new general ideas: this time the concepts of automorphism, homomorphism, and a general discussion of abstract equivalence and congruence relations.

The authors express the belief that "for many students, the value of algebra lies in its applications to other fields: higher analysis, geometry, physics, and philosophy." This belief is reflected in many places throughout the book; but one of the most striking examples is found in their treatment of matrices and determinants (Chapters VII–X), which is so arranged as to be an excellent foundation for a later study of Hilbert and Banach spaces. Before matrices are men-

tioned at all the concepts of vector space, linear dependence, dimension (including infinite-dimensional spaces), inner product, euclidean vector space, normal orthogonal base are considered. Then matrices are introduced as arrays which describe linear transformations of a vector space (relative to some basis); and various properties of matrices are defined relative to the central concept of vector space. Linear groups are introduced as automorphisms of vector spaces. The full linear, the affine, the orthogonal, and the euclidean groups are studied. The problem of classifying quadratic forms relative to these groups is considered.

To clarify the two possible interpretations of change of basis the authors introduce the terms *alibi* (the point moves elsewhere) and *alias* (the point is renamed). This distinction pays rich dividends in the later discussions of various kinds of equivalence and canonical forms. While treating groups of transformations it is natural to discuss invariants under these groups. At this stage the authors insert a discussion of the place of canonical forms in invariant theory. In the reviewer's opinion these little general discussions that appear from time to time are one of the most valuable contributions of the book.

The last of these four chapters opens with a discussion on ranks of matrices and a treatment of the elementary properties of determinants. The authors' interest in applications of algebra to other fields is evidenced here by an interpretation of determinants as volumes. The chapter closes with a survey of the theory of similarity of matrices. Their development does not cover the case of a matrix whose minimum equation has repeated roots, but what they do give illustrates the general ideas and includes many classes of matrices important in other fields.

The recent wave of interest in lattice theory and partially ordered sets is reflected in Chapter XI. The first part of this chapter is a brief treatment of Boolean algebras; then lattices are introduced as a generalization. Chapter XII is devoted to transfinite arithmetic.

In Chapters XIII and XIV enough material is included to give the student the flavor of the theory of rings and fields, but for the real substance of these topics he will have to look elsewhere. The treatment of fields goes far enough to serve as a basis for the final chapter which gives Artin's development of the Galois theory of equations. Solvable groups are studied, and the insolvability by radicals of the general quintic equation is proved.

A very important feature of the book is the inclusion of many exercises. Most of these are illustrative in nature, but a goodly number involve useful extensions of the theory as developed in the text.

The authors' preface includes a summary of the purposes and contents of each chapter, and also indicates several levels of courses in which the book is designed to be used as a text. The most elementary of these is a one year undergraduate course based on only high school algebra. Provision is made for a semester course covering the tools used in physical applications. Finally, a semester course on abstract algebra can be carved out by selecting suitable chapters. The reviewer recommends the use of the book as a text in any of these ways.

The format is pleasing and the text seems to be remarkably free from typographical errors and minor slips.

R. M. THRALL

NOTES

The following one hundred doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1941 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

J. C. Abbott, Notre Dame, August, *The projective theory of non-euclidean geometry.*

R. L. Anderson, Iowa State, June, minors in statistics and economics, *Serial correlation in the analysis of time series.*

Elizabeth S. Arnold, California (Berkeley), December, *On certain projective properties of the configurations of m general hyperplanes in hyperspace.*

K. J. Arnold, Massachusetts Institute of Technology, June, minor in economics, *On spherical probability distributions.*

J. D. Bankier, Rice, June, *Arithmetical continued fractions.*

C. B. Barker, California (Berkeley), December, *The Lagrange multiplier rule for two dependent and two independent variables.*

J. H. Bell, Wisconsin, March, *Topics related to the factorization of matrices.*

T. J. Benac, Yale, June, *The associativity condition for linear associative algebras.*

W. D. Berg, Iowa, August, *Theorems on certain type-A difference-equation graduations.*

F. C. Biesele, Texas, June, *Substitutes for the commutative law in the theory of semi-groups.*

D. H. Blackwell, Illinois, June, minor in physics, *Some properties of Markhoff chains.*

E. E. Blanche, Illinois, June, major in statistics, minor in economics, *A systematic analysis of frequency distributions by the Edgeworth method.*

E. L. Buell, Massachusetts Institute of Technology, June, minor in

physics, *Solution of some problems in plane stress using functions of a complex variable.*

P. B. Burcham, Northwestern, June, *Certain inclusion relations in the domain of Hausdorff methods of summation.*

I. W. Burr, Michigan, June, *Cumulative frequency functions.*

Ethel B. Callahan, Columbia, April, *Some properties of the system of trajectories defined by equations* $\bar{X} = \Phi(X, Y)$, $\bar{Y} = \Psi(X, Y)$.

P. W. Carruth, Illinois, June, minor in astronomy, *Valuations with given residue class field and value group.*

A. B. Carson, Chicago, August, *An analogue of Green's theorem for multiple integral problems in the calculus of variations.*

Mary D. Clement, Chicago, August, *Loci of osculants of asymptotic plane sections of a surface.*

A. C. Cohen, Jr., Michigan, June, *Estimation of parameters in truncated Pearson frequency distributions.*

Esther Comegys, Radcliffe, June, *Sufficient conditions for applicability of Riemannian spaces.*

E. L. Crow, Wisconsin, June, *The expansion problem associated with an ordinary differential equation of the first order which is quadratic in the parameter.*

Benjamin Epstein, Illinois, June, minor in physics, *On a certain class of transforms.*

G. E. Forsythe, Brown, June, *Summability of random variables.*

David Gilbarg, Indiana, June, minor in physics, *On the structure of the group of* $\mathfrak{P}$*-adic* 1*-units.*

F. G. Gravalos, Harvard, June, *The algebraic integrals of Hill's equations.*

P. E. Guenther, Harvard, June, *On a canonical form for the linear homogeneous q-difference equations.*

W. W. Gutzman, Iowa, August, *On the distribution of means of middle items.*

Margaret M. Hansman, Illinois, February, minors in physics and statistics, *A geometric investigation into metabelian groups generated by four elements of order P.*

W. J. Harrington, Cornell, June, *A study of certain functions auxiliary to Brun's method in number theory.*

J. G. Herriot, Brown, June, I. *Cesàro summability of ordinary double Dirichlet series.* II. *Nörlund summability of double Fourier series.*

G. P. Hochschild, Princeton, June, *Semi-simple algebras and generalized derivations.*

F. E. Hohn, Illinois, October, minor in physics, *Curves on Cayley's dianodal surface.*

C. T. Hsu, Columbia, June, *Tests of certain statistical hypotheses concerning bivariate normal populations.*

H. B. Huntington, Princeton, June, major in mathematical physics, *Mechanisms for self-diffusion in a copper lattice.*

L. W. Johnson, Princeton, January, *A linear algebraic theory of complexes.*

R. E. Johnson, Wisconsin, June, *Rings of infinite matrices and polynomial rings.*

H. F. S. Jonah, Purdue, June, minor in mathematical physics, *Congruences connected with the solution of a certain diophantine equation.*

Irving Kaplansky, Harvard, June, *Maximal fields with valuations.*

Lois Kiefer, Illinois, June, minor in astronomy, *A radial analysis of the structure of the Milky Way in Auriga, based on star counts and color excesses.*

E. R. Kolchin, Columbia, September, *On the exponents of differential ideals.*

David Krabill, Ohio State, August, *Matrices whose elements are functions of one variable.*

H. N. Laden, Pennsylvania, June, *An application of the classical orthogonal polynomials to the theory of interpolation.*

J. P. LaSalle, California Institute of Technology, June, minor in physics, *Pseudo-normed linear sets over valued rings.*

S. J. Lawwill, Cincinnati, June, *Concerning harmonic functions especially with reference to the theory of approximation.*

Jeanne S. LeCaine, Radcliffe, June, *On the generalized gamma function and its analogue for q-difference equations.*

H. L. Lee, Duke, June, *Power sums of polynomials in a Galois field.*

Joseph Lehner, Pennsylvania, June, *A partition function connected with the modulus five.*

R. J. Levit, California (Berkeley), May, *Characterizations of fields as systems of single composition.*

C. B. Lindquist, Wisconsin, September, *Rectangular isotropic and anisotropic plates under forces in their planes.*

D. B. Lloyd, Catholic University, February, minors in physics and education, *Some properties of rational quintic equations.*

L. A. Lorch, Cincinnati, June, *Some problems in Borel summability of Fourier series.*

Leonard McFadden, Brown, October, *Absolute Nörlund summability.*

Rhoda Manning, Stanford, June, minor in biochemistry, *On the derivatives of the sections of bounded power series.*

A. E. Marston, California (Berkeley), December, *A linear elliptic partial differential equation whose leading coefficients may vanish at a point.*

F. L. Martin, Chicago, August, *Integral domains in quartic fields.*

Herman Meyer, Chicago, June, *Polynomial approximations to functions defined on abstract spaces.*

E. J. Mickle, Ohio State, June, *Hamiltonian and quasi-Hamiltonian functions associated with double integral variation problems.*

D. D. Miller, Michigan, February, *Extension and reduction theorems for certain types of continuous transformations.*

D. S. Miller, Cornell, September, *Some properties of Carathéodory and Gillespie linear measure.*

G. T. Miller, Purdue, June, minors in statistics and mathematical physics, *Analytic continuation of functions defined by factorial series.*

Harlan C. Miller, Texas, June, *On compact unicoherent continua.*

Philip Newman, Columbia, June, *The geometry of the planar* (2, 2) *connex.*

E. H. Nicholson, Washington University, June, minor in physics, *On the degree of approximation in some convergence theorems concerning derivatives of the mapping function in conformal mapping.*

E. N. Nilson, Harvard, June, I. *Interpolation and approximation to an analytic function by functions analytic and bounded in a region.* II. *Approximation to a harmonic function by functions harmonic and bounded in a region.*

C. G. A. Nordling, Massachusetts Institute of Technology, June, minor in mechanical engineering, *Solution by polynomial approximation of the integral equation for the circulation around an airfoil.*

Sister Jeanette Obrist, Catholic University, February, minors in physics and botany, *A problem arising from the special symmetric correspondence C_2 set up by the rational quartic curve with two cusps.*

R. E. O'Connor, Harvard, June, *Representation of integers by power-products of two real numbers.*

J. W. Odle, Michigan, February, *Non-alternating and non-separating transformations modulo a family of sets.*

O. G. Owens, California (Berkeley), May, *An explicit formula for the solution of the ultrahyperbolic equation in four variables.*

A. S. Peters, New York, June, minor in physics, *On a simply supported-free half-plane plate and a clamped infinite-sector plate.*

Anatol Rapoport, Chicago, December, *Construction of non-abelian fields with prescribed arithmetic.*

G. E. Reves, Cincinnati, June, *On the absolute convergence of double Fourier series.*

C. E. Rickart, Michigan, June, *Integration in a convex, linear topological space.*

L. A. Ringenberg, Ohio State, March, *On functions of intervals.*

F. E. Satterthwaite, Iowa, August, *Developments on the theory of Chi-square.*

L. J. Savage, Michigan, June, *The application of vectorial methods to the study of distance spaces.*

Ivor Schilansky, Michigan, June, *On generalized zeta functions and their associated lattice point problems.*

W. M. Scott, Michigan, February, *On matrix algebras over an algebraically closed field.*

D. M. Seward, Duke, June, minor in physics, *Harmonic continuation in space.*

Ernst Snapper, Princeton, January, *Structure of linear sets.*

C. D. Solin, Toronto, June, *Positive ternary quadratic forms of determimant* $\leqq 200$, *and positive ternary quadratic forms in genera of one class.*

R. H. Sorgenfrey, Texas, June, *Concerning triodic continua.*

A. H. Sprague, Princeton, January, *Surfaces whose lines of curvature are nets R, and their transformations.*

C. F. Strobel, Illinois, June, minor in physics, *The quadrilinear form* (1, 1, 1, 2).

R. L. Swain, Texas, June, I. *Proper and reductive-transformations.* II. *Continua obtained from sequences of simple chains of point sets.* III. *Distance axioms in Moore spaces.* IV. *Linear metric space.* V. *A space in which there may exist uncountable convergent sequences of points.*

C. J. Thorne, Iowa State, June, minor in physics, *The approximate solution of linear differential equations by the use of functionals.*

L. V. Toralballa, Michigan, June, *The sum of the values of a rational function of s variables over the set of all the n-partite permutational partitions of a given positive integer.*

W. R. Transue, Lehigh, October, minor in physics, *Contributions to the theory of subharmonic functions.*

J. R. Vatnsdal, Michigan, February, *Minimal variance and its relation to efficient moment tests.*

L. I. Wade, Jr., Duke, June, minor in philosophy, *Certain quantities transcendental over the field* $GF(p^n, x)$.

Muriel Wales, Toronto, June, *Theory of algebraic functions based on the use of cycles.*

Dzung-shu Wei, Iowa, February, *Necessary and sufficient condi-*

tions that regression systems of sums with elements in common be linear.

William Wernick, New York, June, minor in physics, *Complete sets of logical functions.*

R. L. Westhafer, Ohio State, August, *Singular solutions of ordinary differential equations of the first order.*

G. W. Whitehead, Jr., Chicago, June, *Homotopy properties of the real orthogonal groups.*

P. M. Whitman, Harvard, June, *Free lattices.*

W. F. Whitmore, California (Berkeley), May, *Convergence theorems for functions of two complex variables.*

W. D. Wray, Cornell, June, *Some applications of uniformity trials.*

Paul Young, Ohio State, August, *On the approximation of functions by integral means.*

Professor J. L. Gibson of the University of Utah has retired with the title professor emeritus.

Assistant Professor Max Astrachan of Antioch College has been promoted to an associate professorship and made chairman of the department of mathematics.

Dr. Alfred Basch of Holy Cross College has accepted a position at the College of Paterson, Paterson, New Jersey.

Assistant Professor T. C. Benton of Pennsylvania State College has been promoted to an associate professorship.

Associate Professor J. E. Davis of Drexel Institute of Technology has been promoted to a professorship.

Associate Professor M. A. Hill of the University of North Carolina has been promoted to a professorship.

Assistant Professor Nathan Jacobson of the University of North Carolina has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan is at the University of Wisconsin for the second semester of this current academic year.

Dr. A. E. Marston of the University of California has accepted a position at Compton Junior College, Compton, California.

Associate Professor F. C. Ogg of Bowling Green State University has been promoted to a professorship.

Dr. Arthur Rosenthal has been appointed to a lectureship at the University of New Mexico.

Professor M. H. Stone has been appointed chairman of the department of mathematics at Harvard University, succeeding Professor J. L. Walsh.

Assistant Professor Antoni Zygmund of Mount Holyoke College has been appointed visiting assistant professor at the University of Michigan for the second semester of the current academic year.

Mr. C. L. Carroll of the Georgia School of Technology is on leave at the University of North Carolina.

Assistant Professor B. E. Gatewood of Louisiana Polytechnic Institute is on leave and with MacDonnell Aircraft Corporation.

Professor W. R. Hutcherson of Berea College is on leave at Brown University.

Professor H. W. March of the University of Wisconsin is on leave of absence.

Professor F. W. Owens of Pennsylvania State College is on leave of absence.

Professor Mary E. Sinclair of Oberlin College is on leave of absence.

Dean T. M. Putnam and Assistant Professor B. C. Wong of the University of California are on leave of absence.

Professor H. C. Carver, Assistant Professor P. S. Dwyer, Professor C. E. Love, and Associate Professor J. A. Nyswander of the University of Michigan are on leave of absence.

The following appointments to instructorships are announced: Massachusetts Institute of Technology: Dr. J. P. LaSalle; University of Wisconsin: Dr. R. E. Johnson, Dr. R. L. Swain.

Miss Frances Hardcastle of Stocksfield-on-Tyne, England, died December 26, 1941. She had been a member of the Society since 1894.

Professor R. L. Charles of Franklin and Marshall College died December 13, 1941, at the age of fifty-six years.

Professor W. V. N. Garretson of Oklahoma Agricultural and Mechanical College died January 17, 1942, at the age of sixty-five years.

Dr. Robert Henderson died February 16, 1942. Until his retirement he had been vice president of Equitable Life Assurance Society of the United States. Dr. Henderson joined the American Mathematical Society in 1910 and served on its Board of Trustees for many years. A more complete notice will appear later in this Bulletin.

Dr. E. R. van Kampen of Johns Hopkins University died February 11, 1942.

Dr. E. J. Maurus, who was professor of mathematics at the University of Notre Dame from 1897 to 1939, died November 26, 1941, at the age of sixty-nine years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

141. A. A. Albert: *The radical of a non-associative algebra.*

The multiplications of an algebra A generate a corresponding associative polynomial algebra $T(A)$. Every ideal B of A determines a corresponding ideal S of $T(A)$ and it is shown that $T(A-B)$ is equivalent to $T(A-S)$. If H is the radical of $T(A)$ the set AH is an ideal of A and $A-AH$ is either a zero algebra Z, a semi-simple algebra G (direct sum of simple algebras), or a direct sum $Z \oplus G$. If $A-AH=Z$ the algebra A is not homomorphic to a semi-simple algebra. For all other algebras the radical is defined to be the minimal ideal N such that $A-N$ is semi-simple. Then $N=AH$ if $A-AH$ is semi-simple. Otherwise $Z=N-AH$, the quantities of N are the quantities in the cosets which make up Z. If A and A_1 are isotopic algebras with unity quantities, their radicals $N=AH$ and $N_1=A_1H_1$ are isotopes and the difference algebras $A-N$ and A_1-N_1 are isotopes. An example is given of an algebra A with a unity quantity whose radical, with respect to our definition, is a field. (Received February 12, 1942.)

142. B. H. Arnold: *Rings of transformations of certain vector spaces.* Preliminary report.

Eidelheit (Studia Mathematica, vol. 9 (1940), pp. 97–105) showed that the algebraic properties of the ring of all continuous linear transformations of a real Banach space into itself characterizes the Banach space up to an isomorphism. Mackey has extended this result to a more general class of spaces. In this paper the author shows what algebraic properties of the ring correspond to certain properties of the space, such as completeness, reflexivity, reflexivity of the completion, or being the conjugate of some space. Eidelheit's result is extended to the complex case and a set of conditions is given which are necessary and sufficient that an abstract ring be the ring of all continuous linear transformations of some space into itself. (Received March 3, 1942.)

143. Emil Artin and Peter Scherk: *On the sum of two sets of integers.*

Let A, B, respectively, be sets of non-negative integers a, b. Let C be the set of all integers of the form $a+b$. Let $A(x)$, $B(x)$, $C(x)$ denote the number of positive inintegers of the sets less than or equal to x. A few months ago, H. B. Mann succeeded in proving: If $0 \subset A$ and $0 \subset B$, and if $C(n)<n$, then $C(n)/n \geqq \min_{x=1,\ldots,n} (A(x)+B(x))/x$. Applying and simplifying Mann's method, the authors have proved: If $C(n)<n$, then $C(n)-C(n-m)=A(m-1)+B(m-1)+Z_m(n)$, for a suitable $m \notin C$

with $0 < m \leqq n$; $Z_m(n)$ denotes the number of decompositions of n of a certain type (hence $Z_m(n) \geqq 0$). If $0 \subset A$, $0 \subset B$, this result contains various estimates of $C(n)$ including that by Mann. If $0 \subset A$ a result of A. S. Besicovitch's follows (Journal of the London Mathematical Society, vol. 10 (1935), pp. 246–248). If $0 \subset\subset A$, $0 \subset\subset B$, new estimates of $C(n)$ follow immediately. (Received March 14, 1942.)

144. S. P. Avann: *A numerical condition for modularity of a finite lattice.*

Let $U(a)$ be the number of elements other than a which are covered by elements covering a, an element of a finite lattice L in which the Jordan-Dedekind chain condition holds; let $V(a)$ be the number of elements other than a which cover elements covered by a. Then a necessary and sufficient condition that L be modular is that $U(a) = V(a)$ for every $a \in L$. Finiteness of L is required for sufficiency. Hence L may be tested for modularity by counting coverings or bonds in the Hasse diagram of L. (Received March 14, 1942.)

145. S. P. Avann: *On complete classes of projective quotients in a modular lattice.*

Let L be a finite modular lattice in which Q is a complete class: $[a_1/b_1, a_2/b_2, \cdots, a_r/b_r]$ of projective quotients defined by the equivalence relation of projectivity. Denote the set of numerators a_i by A and the set of denominators b_i by B. Then the following conditions are equivalent: (1) A and B are transposed quotient sublattices of L, (2) the meet a_0 of the a_i is in A and to each $a_i \neq a_0$ belongs a unique denominator b_i in B, (2′) the join b of the b_i is in B and to each $b_i \neq b$ belongs a unique numerator a_i in A, (3) $a_0 \in A$ and $b \in B$, (4) uniqueness of denominators and numerators. If the quotient sublattices of Q are complemented then (1), (5) $a_0 \in A$, (5′) $b \in B$, (6) uniqueness of denominators, (6′) uniqueness of numerators, are equivalent conditions. (Received March 14, 1942.)

146. Reinhold Baer: *Radical ideals.*

An ideal is termed a radical ideal, if it is a two-sided nil ideal modulo which there do not exist nilpotent right-ideals different from 0. Every ring possesses at least one radical ideal; and both the cross-cut and the sum of all the radical ideals are themselves radical ideals which are called the lower and the upper radical, respectively. It is possible that lower and upper radical are different and that neither of them is nilpotent. If every right ideal, not 0, in the quotient ring modulo the lower radical contains a smallest right ideal different from 0, then upper and lower radical are equal; and if this condition is satisfied by every quotient ring, then a finite or transfinite power of the radical ideal is 0. As an application of this theory a generalization of a theorem of C. Hopkins is proved. (Received March 12, 1942.)

147. Richard Brauer and Hsio-Fu Tuan: *Some remarks on simple groups of finite order.* Preliminary report.

Using the theory of group characters, some lemmas on simple groups of finite order are obtained. For instance, a group G of order $3 \cdot p^a \cdot q^b$ (p, q primes) can only be simple if G has one of the orders 60 or 168. It can be shown that the list of simple groups (cf. L. E. Dickson, *Linear Groups*) is complete up to the order 10,000. (Received March 6, 1942.)

148. Leonard Carlitz: *On the reciprocal of certain types of Hurwitz series.*

In the first part of this paper previous results (Duke Mathematical Journal, vol. 9 (1942)) are extended to series $\sum A_m t^{p^m}/F_m$, where the A_m are polynomials of a more general type. In the second part of the paper the case $\sum A_m t^m/G_m$ is treated. (Received March 6, 1942.)

149. R. F. Clippinger: *Matrix products of matrix powers.*

If $A_1, A_2, \cdots, A_m$ are given matrices of constants, the closure of the set of products of arbitrary positive powers of the matrices $\overline{A}_1=\exp A_1, \overline{A}_2=\exp A_2, \cdots, \overline{A}_m=\exp A_m$ is identical with the closure of the set of solutions for positive t of the matrix differential equation: $dy/dt=Y(\rho_1(t)At_1+\rho_2(t)A_2+\cdots+\rho_m(t)A_m)$, $Y(0)=1$, where $\rho_1(t), \rho_2(t), \cdots, \rho_m(t)$ are arbitrary, non-negative, measurable, bounded functions of t. (Received March 19, 1942.)

150. R. P. Dilworth: *The partition lattice.*

In a previous paper (abstract 48-1-11) a dependence relation for elements of rank k in a Birkhoff lattice was defined. In this paper it is shown that if this dependence relation is applied to the elements of rank two in a Boolean algebra, the closed subsets form a lattice isomorphic to the partition lattice. (Received March 9, 1942.)

151. F. A. Ficken: *On Rosser's generalization of Euclid's algorithm.* Preliminary report.

Let $X_1, \cdots, X_n$ be a set of vectors in an euclidean space. The set of independent vectors $Y_1, \cdots, Y_m$ is a greatest common factor (G.C.F.) of the X's if each $X_i=\sum p_{ji}Y_j$ and each $Y_k=\sum q_{kl}X_l$, where the p's and q's are integers. In a minimal G.C.F., the lengths of the Y's satisfy certain minimal conditions. Full definitions and a number of important applications may be found in Duke Mathematical Journal, vol. 9 (1942), pp. 59–95. Rosser has given there an algorithm for finding G.C.F.'s and all minimal G.C.F.'s. By examining the termination of the algorithm, he establishes several properties of minimal G.C.F.'s. In the present paper, this examination is conducted by producing explicit lower bounds for $(\sum t_iX_i)^2$, $(i=1, \cdots, n; n=2, 3, 4, 5)$. These bounds furnish simplified proofs for the bulk of the results and confirm certain of the conjectures stated in Rosser's paper. (Received March 6, 1942.)

152. J. S. Frame: *Double coset matrices and group characters.*

This paper extends to the characters of the irreducible components of an intransitive permutation group a theorem proved earlier by the author (this Bulletin, vol. 47 (1941), p. 459) for the degrees of the irreducible components of a transitive permutation group. To the element γ of G corresponds the $n\times n$ permutation matrix $R(\gamma)$ with the $n^t\times n^t$ transitive constituents $R^t(\gamma)$ which contain the irreducible representation Γ_i just μ_i^t times. To each transitive constituent corresponds a subgroup H^t of order h^t leaving its first symbol fixed. The matrices V permutable with G break up into submatrices V^{ts} linearly dependent on μ^{ts} double coset matrices V_α^{ts}, each associated with a double coset $H_\alpha^{st}=H^s\gamma_\alpha H^t/h_\alpha^{st}$ formed from the given subgroups. With the structure coefficients $c_{\alpha\beta\eta}^{rst}$ which appear in the product formula $H_\alpha^{rs}H_\beta^{st}/h^s=\sum_\eta c_{\alpha\beta\eta}^{rst}H_\eta^{rt}$, with the constants $k_{\eta\lambda}^t$ which tell how many conjugate elements from the class C lie in the double coset H_η^{st}, and with a set of arbitrary parameters a_λ, a matrix

$K^{st}=\sum_{\lambda,\eta}c_{\alpha\beta\gamma}^{\lambda\mu}k_{\eta\lambda}^{t}a_{\lambda}$ is formed. Its determinant $|K^{st}|$ will factor into linear factors $\chi_i=\chi_i(\sum a_\lambda C_\lambda)$ which give the characters of the irreducible components Γ_i common to two constituents $R^s(\gamma)$ and $R^t(\gamma)$. The theorem is: $(n^t)\mu^{st}|K^{st}|=P_{st}\bar{P}_{st}\prod_i(\chi_i)\mu_i$, where $\mu_i=\mu_i^s\mu_i^t$, $\mu^{st}=\sum\mu_i$, and P_{st} is an algebraic integer. An illustrative example is given. (Received March 5, 1942.)

153. R. D. James: *On Euler's conjecture.*

It was conjectured by Euler that every integer $N\equiv 2 \pmod 4$, $N\geqq 10$, could be represented as a sum of two primes each congruent to 1 (mod 4). However even the weaker statement that every such integer is a sum of two integers which have all their prime factors congruent to 1 (mod 4) has not yet been proved. In a previous paper (Transactions of this Society, vol. 43 (1938) pp. 296–302) the author proved that every sufficiently large integer $N\equiv 2 \pmod 4$ is a sum of two integers which have all except possibly two of their prime factors congruent to 1 (mod 4). This result is improved in the present paper as indicated in the following theorem: Every sufficiently large integer $N\equiv 2 \pmod 4$ is a sum of two integers which either have at most six prime factors all congruent to 1 (mod 4) or else have exactly two prime factors congruent to 3 (mod 4) and at most three prime factors congruent to 1 (mod 4). The method of proof is a general one and applies to any infinite sequence of primes for which $\sum p^{-1}\log p=\mu\log x+O(1)$, where μ is any positive number. (Received March 20, 1942.)

154. R. E. Johnson: *On structures of infinite modules.*

Let Q be a principal ideal ring containing the field P, and Ξ be a P-module which has Q as an operator domain. It is assumed that Ξ has a countable P-basis. Then the problem studied is under what conditions there exist elements $\xi_1, \xi_2, \cdots$ in Ξ such that $\Xi=Q\xi_1+Q\xi_2+\cdots$, this last being a supplementary sum. Necessary and sufficient conditions are found for this to be the case. If $Q/(m)$, m not a unit of Q, is taken as the operator domain of Ξ, then there always exist elements $\xi_1, \xi_2, \cdots$, of Ξ such that $\Xi=Q/(m)\xi_1+Q/(m)\xi_2+\cdots$. Now let $\mathfrak{M}_\omega$ be the matrix algebra of order type ω over P with finitely nonzero columns, and Ξ be the P-module composed of the columns of these matrices. Then an element A of $\mathfrak{M}_\omega$ can be transformed into a reduced form in case $\Xi=P[A]\xi_1+P[A]\xi_2+\cdots$. (Received March 18, 1942.)

155. H. F. S. Jonah: *Congruences connected with the solution of a certain Diophantine equation.*

The paper begins with a function and functional equation used by Maier (Mathematische Annalen, vol. 104 (1931), pp. 745–769) and generalizes both. Then by means of the calculus of residues the author obtains a quadratic functional equation of a function closely related to the $h_n^{(a,b)}(x)$ of Vandiver (Annals of Mathematics, (2), vol. 27 (1926), pp. 171–176). Using this functional equation it is possible to transform the general congruences used by Vandiver in his work on Fermat's last theorem. (Received March 18, 1942.)

156. Joseph Lehner: *On certain irrational modular equations.*

The author obtains by simple group-theoretic considerations the modular equation connecting $\eta(\tau)/\eta(\tau/p)$ and $\eta(p\tau)/\eta(\tau)$, where $\eta(\tau)=\exp(\pi i\tau/12)\cdot\prod_1^\infty(1-\exp(2\pi in\tau))$ and $p=5, 7, 13$. These are the only three primes p for which an equation of degree p exists, this fact being connected with the genus of the subgroup

$\Gamma_0(p)$ of the full modular group. The equations for $p=5, 7$ were obtained by Watson (Crelle, 1937) by more complicated methods; they are important in the theory of Ramanujan's congruences concerning the unrestricted partition function, $p(n)$. (Received February 21, 1942.)

157. F. A. Lewis: *Generators of permutation groups simply isomorphic with* $LF(2, p^n)$.

Generators of any permutation group of the class are obtained by a slight extension of an argument used in a recent paper (this Bulletin, vol. 47 (1941), p. 631). (Received February 20, 1942.)

158. Knox Millsaps: *Abstract polynomials in non-abelian groups.*

After an abstract calculus of finite differences is defined, functional definitions of a monomial and a polynomial for elements of the central subgroup or elements of the group as increments are given. The theorem on the homogeneity of a polynomial is proved for central and arbitrary differences; for central differences the difference being a function of the increment alone implies the difference is a monomial; the independence of the central difference of polynomials and the unique decomposition for the abelian valued case are made to depend on the product of a Vandermonde determinant and a finite product of binomial coefficients; and these last results can be extended to the non-abelian valued case for certain free groups. The theory is essentially a generalization of the work of Van der Lijn on abstract polynomials in abelian groups. (Received March 4, 1942.)

159. Ivan Niven: *On matric polynomials.*

Let B be a matrix with elements from any field, and let P be a scalar polynomial in B with coefficients from the field. Then the index of P cannot exceed the index of B. Furthermore, B is expressible as a polynomial in P if and only if the index of P equals the index of B. A new method is offered for finding those solutions X of $X^m=A$ which are polynomials in A, the elements of A being from an algebraically closed field with characteristic 0 (cf. M. H. Ingraham, *Rational methods in matrix equations*, this Bulletin, vol. 47 (1941), p. 64, Theorem 3). If A is non-derogatory every solution of $X^m=A$ is a polynomial in A, and the equation has no solutions if and only if 0 is a repeated root of the minimal equation of A. (Received March 17, 1942.)

160. Rufus Oldenburger: *Expansions of quadratic forms.*

A quadratic form Q is ordinarily given as a sum $a_{ij}x_ix_j$ of power products of the variables. Although this is a sum of products of linear forms, this is not in general the most economical way of writing Q as such a sum. The minimum τ such that Q is a sum of τ products of linear forms is $r-\sigma$, where r is the rank of Q, and σ is the index of Q introduced elsewhere by the author. This result is valid for fields with characteristic not 2. (Received February 12, 1942.)

161. Rufus Oldenburger: *Repeated linear factors of forms.*

With a form F of degree n there are associated 2-way displays, $A_0, A_1, \cdots, A_n$ of the symmetric matrix A of coefficients of F, used by W. Mayer, Thrall and Chandler, F. L. Hitchcock, and others. The form F has a linear factor L repeated r times if and only if L^{n-r+1} is a covariant directly constructible from A_{n-r+1}. If n is even the coefficients in L^{n-r+1} can be expressed simply as higher-dimensional determinants. (Received February 20, 1942.)

162. Rufus Oldenburger: *The index of a quadratic form for an arbitary field.*

The classical index of a quadratic from Q is defined in terms of the order of elements in the real field. In the present paper a definition of the index of Q is given which is valid for any field, and which reduces in the real case to the index of Q or $-Q$. It is proved that a form Q with rank r and index σ has a splitting $G+H$, where G has index σ and rank 2σ, whereas H has index 0 and rank $r-2\sigma$. A second form Q' with a corresponding splitting $G'+H'$ is equivalent under a nonsingular linear transformation to Q if and only if the rank and index of Q equal the rank and index of Q', and H is equivalent to H'. A form H has index 0 if and only if H is a nonzero form. Thus the problem of the equivalence of quadratic forms reduces to that of the equivalence of nonzero forms. Such forms have been studied by Siegal, Weyl, Dickson, Legendre, and many others. (Received February 13, 1942.)

163. M. A. Sadowsky: *Approximation by rational sequences.*

A certain graphical principle is employed to select the best approximations to a real number. It is shown that this process leads to the consecutive convergents of the continued fraction development. (Received March 20, 1942.)

164. R. D. Schafer: *Alternative algebras over an arbitrary field.*

The results of M. Zorn concerning simple alternative algebras are incomplete for algebras over modular fields of characteristics two or three. This paper presents a unified treatment of such algebras over an arbitrary field. Define a quaternion algebra Q over F as in A. A. Albert's *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, p. 145. Then an alternative, but not associative, algebra A over F is central simple (simple for all scalar extensions) if and only if A is a Cayley-Dickson algebra $A=Q+gQ$, in which multiplication is defined by $(q_1+gq_2)(q_3+gq_4)=(q_1q_3+\delta q_4\cdot q_2S)+g(q_1S\cdot q_4+q_3q_2)$, where $g^2=\delta\neq 0$ in F and S is the involution $z\leftrightarrow zS=t(z)-z$, the principal function of A being $z^2-t(z)\cdot z+n(z)$. Moreover A is a central division algebra if and only if Q is a division algebra and there exist no λ, μ, ρ, σ in F such that $\delta=\lambda^2+\lambda\mu-\beta\mu^2-\gamma\rho^2-\gamma\rho\sigma+\beta\gamma\sigma^2$. (Received March 20, 1942.)

165. M. F. Smiley and W. R. Transue: *Applications of transitivities of betweenness in lattic theory.*

This paper is an expansion of our note *Metric lattices as singular metric spaces* (this Bulletin, abstract 48-1-20) which was to appear in this Bulletin. The properties of betweenness studied in the original note are generalized and applied to the characterization of modular lattices and of arbitrary lattices. The proofs of the extensions of our original results are based in part on certain of the results of the paper *Transitivities of betweenness* by Everett Pitcher and M. F. Smiley (forthcoming in the Transactions of this Society). (Received February 27, 1942.)

166. R. M. Thrall: *On the decomposition of modular tensors.*

Let k be any field, and denote by A any element of the full linear group $G(n, k)$ of n-rowed k-matrices. The mapping $A\rightarrow\pi^m(A)=A\times\cdots\times A$ (Kronecker direct product with m factors A) is a representation of $G(n, k)$ as linear transformations on the space V_m of all tensors of rank m. For k of characteristic zero the theory of the decomposition of V_m into invariant subspaces is classical. A key tool of the develop-

ment of this theory is use of the relationship between V_m and the symmetric group of degree m. For k of characteristic p the classical theory is valid only for $m<p$. In the present paper the decomposition of V_m is obtained for $p \leq m < 2p$. Further progress on the decomposition problem will likely have to await the development of the theory of modular representations of the symmetric groups of degree $m \geq 2p$; the measure of difficulty here being the power of p which divides $m!$. (Received March 26, 1942.)

167. Bernard Vinograde: *Split rings and their representation theory.*

Suppose a ring R, with minimum condition on left ideals, can be split into a direct sum, $R=R^*+N$, of a semi-simple ring and the radical. This splitting property of R is equivalent to the existence of a certain set of noncommutative fields in R. Let R have a unit, and let V be a commutative group possessing R in its left operator domain and a finite composition series with respect to R. Then V is a direct sum of modules, each over one of the noncommutative fields. Thus V gives rise to a composite representation module for its homomorphism ring. In the case $V=R$ the resulting representation of R displays the essential structure of R^* and N, and may be considered a natural extension of the Wedderburn theorem for simple rings to the class of split rings with unit and finite composition series. (Received March 20, 1942.)

168. R. W. Wagner: *Expressibility relations for bilinear operations.* Preliminary report.

If xy is a multiplication operation for a vector space over a field, an expression like $(a((xb)(cy)))d$ is also a multiplication for the vector space. The operation x^*y is called simply expressible in the operation xy if x^*y is a finite iteration of the original operation. The operation x^*y is called strongly expressible if it is a linear combination of weakly expressible operations. Necessary and sufficient conditions are found for both kinds of expressibility. These conditions involve the enveloping algebra of the algebra formed by the original multiplication. Mutual expressibility, both simple and strong, is an equivalence relation and differs from isotopy. Reducibility is a property common to mutually expressible algebras. Associativity, commutativity, and the existence of nilpotents are properties which need not be preserved. (Received March 18, 1942.)

Analysis

169. Dorothy L. Bernstein and S. M. Ulam: *On the problem of completely additive measure in classes of sets with a general equivalence relation.*

The problem of finding a necessary and sufficient condition for the existence of a finitely additive measure in a class of sets, with the property that equivalent sets have equal measure, has been solved by Tarski (Fundamenta Mathematicae, vol. 31 (1938), pp. 47–66). This paper considers the existence of a completely additive measure in a class K which is a Borel field over a given sequence $\{A_n\}$ of sets and which has the property that $m(X)=m(Y)$ if X is equivalent to Y. Equivalence is defined in the most general sense: Given a division of the sets of K into disjoint classes $A_1, A_2, A_3, \cdots$, sets X and Y are called equivalent if they belong to the same class A_α. It should be noted that even if no requirement for equivalence is made, it is not always possible to find a measure in a Borel field over a given sequence of sets. A necessary and suffi-

cient condition is obtained which involves the characteristic set of real numbers formed from the sequence $\{A_n\}$ and the existence of a homeomorphism of this set with a set of positive outer Lebesgue measure, in which intervals corresponding to equivalent sets have equal outer measure. (Received March 20, 1942.)

170. Lipman Bers: *A property of harmonic functions.*

Suppose that $u(z)=(x+iy)$ is a non-negative harmonic function defined in $|z|<1$. Consider the upper and lower limits, $L(\theta)$ and $l(\theta)$, of $u(z)$ as z goes toward the point 1 along a simple Jordan curve which possesses at 1 a definite direction forming the angle θ, $|\theta|<\pi/2$, with the positive x-direction. (These limits depend only upon θ.) $L(\theta)$ and $-l(\theta)$ are uniformly continuous and convex functions. If $-\pi/2\leqq\theta_1<\theta<\theta_2\leqq\pi/2$, $l(\theta)\leqq\mu L(\theta_1)+(1-\mu)l(\theta_2)\leqq L(\theta)$ and $l(\theta)\leqq\mu l(\theta_1)+(1-\mu)L(\theta_2)\leqq L(\theta)$ where $\mu=(\theta_2-\theta)/(\theta_2-\theta_1)$. If $u(z)$ possesses at 1 a limit in one direction, that is, if $L(\theta_0)=l(\theta_0)$, $L(\theta)$ and $l(\theta)$ are linear functions in the intervals $-\pi/2\leqq\theta\leqq\theta_0$ and $\theta_0\leqq\theta\leqq\pi/2$. If $u(z)$ possesses at 1 limits in two different directions, $u(z)$ possesses at 1 a limit in any direction and $L(\theta)=l(\theta)$ is a linear function. If $u(z)$ is bounded and its Dirichlet integral finite, $L(\theta)$ and $l(\theta)$ are equal to the upper and lower limits of $\tau^{-1}\{\mu[V(\tau)-V(0)]+(1-\mu)[V(0)-V(-\tau)]\}$, $\tau\to+0$, where $\mu=(\pi+2\theta)/2\pi$ and $V(\tau)$ is the indefinite integral of $U(\tau)=u(e^{i\tau})$. (Received March 28, 1942.)

171. F. G. Dressel: *The fundamental solution of the parabolic equation.* II.

The existence of a fundamental solution in an unbounded region for the parabolic equation with variable coefficients is proved in this paper. In addition, several properties of this fundamental solution are demonstrated. (Received February 5, 1942.)

172. E. D. Hellinger and H. S. Wall: *Contributions to the analytic theory of continued fractions.*

The theory of (1) $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$, as developed by Stieltjes, Hamburger, Hellinger, R. Nevanlinna, and others is extended to the case $a_n\neq0$ real and $\Im(b_n)\geqq0$. The nth approximant lies upon a circle $K_n(z)$ contained in $K_{n-1}(z)$, $(\Im(z)>0)$. *Case* I. For $\Im(z)>0$ the circles have only one point in common and (1) converges; the associated J-form which, in contrast with former extensions of this theory, is not hermitian, has a unique resolvent bounded on an "ellipsoid" E. *Case* II. Otherwise, (1) converges for all z (to a meromorphic function of z) if and only if it converges for $z=0$; the J-form has infinitely many resolvents completely continuous on E. The associated power series of (1) represents $f(z)$ asymptotically in $\epsilon\leqq\arg z\leqq\pi-\epsilon$, $\epsilon>0$, if and only if for every n there exists an M_n such that $f(z)$ lies in $K_n(z)$ for z in the sector and $\Im(z)>M_n$. If for $\Im(z)>0$ the value of an analytic function $f(z)$ is in $K_n(z)$ for every n, then $f(z)=\int_{-\infty}^{+\infty}d\phi(u)/(z+u)$, $\phi(u)$ real, bounded, monotone non-decreasing; and (1) represents a function of this form whenever it converges. The moments $\int_{-\infty}^{+\infty}u^n d\phi(u)$ do not in general exist. (Received March 6, 1942.)

173. Fritz John: *The character of solutions of linear partial differential equations.*

Let $L(u)=0$ denote a linear partial differential equation with analytic coefficients for a function $u(x_1,\cdots,x_n)$ (order N or "type" of equation as well as the number n of independent variables are arbitrary). Characteristic directions with respect to L at a point may be defined in the usual manner. Let there be given an analytic k-

parametric family of r-dimensional manifolds $M_{\alpha_1 \cdots \alpha_k}$ in $x_1 \cdots x_n$-space ($r<n$), such that (a) all $M_{\alpha_1 \cdots \alpha_k}$ have the same boundary, and (b) none of their normals is characteristic. Let $d\omega$ be any analytic element of mass on the $M_{\alpha_1 \cdots \alpha_k}$. The main result of this paper states that $\int u d\omega$ over $M_{\alpha_1 \cdots \alpha_k}$ will be an analytic function of $\alpha_1, \cdots, \alpha_k$, whenever u is any solution of class C^N of the equation $L(u)=0$. The analyticity of solutions of linear elliptic equations with analytic coefficients is a special case. It also follows that the values of u cannot be prescribed arbitrarily on any analytic manifold in $x_1 \cdots x_n$-space, which contains a sub-manifold M of lower dimension, which has no characteristic normals. (Received March 12, 1942.)

174. M. L. Kales: *Tauberian theorems.*

Let $T_{m,n} \geqq 0$, $\sum_{n=0}^{\infty} T_{m,n} \equiv 1$, and $\{A_n\}$ be a sequence of real numbers. Under fairly general conditions it is shown that if $\sum_{n=0}^{\infty} T_{m,n} A_n = O(1)$ as $m \to \infty$, and $A_n - A_{n-1} > -\psi(m, n)$ (where the function $\psi(m, n)$ is defined in the statement of the theorem) then $A_n = O(1)$ as $n \to \infty$. This result is then applied to prove the following theorem: Let $\lim_{x \to \infty} g'(x) = \alpha (0 < \alpha \leqq \infty)$. Let $T_{m,n} = (1/\sum_{k=0}^{\infty} \exp\{-g(k)+kg'(m)\}) \exp\{-g(n)+ng'(m)\}$. If $\lim_{m \to \infty} \sum_{n=0}^{\infty} T_{m,n} A_n = A$, and if $A_n - A_{n-1} > -K \exp\{g(n)-g(n-1)-g'(n-1)\}$, then $\lim_{n \to \infty} A_n = A$. The proof of this theorem for the case where α is infinite follows immediately from the first theorem. When α is finite the problem can be reduced to one where an application of one of Wiener's general Tauberian theorems is possible. (Received March 18, 1942.)

175. Meyer Karlin: *Characterization of the extremals of a variation problem of higher order in the plane.* Preliminary report.

A variation problem of second order in the plane, $\int \phi(x, y, y', y'')dx = \min$, gives rise to a quadruply infinite system of curves for its extremals. E. Kasner (this Bulletin, vol. 13 (1907), pp. 289–292) found their differential equation to be of the form: $y^{IV} = Ay'''^2 + By''' + C$, where A, B, C involve x, y, y', y''. With the help of a theorem by A. Hirsch (Mathematische Annalen, vol. 49 (1897), pp. 49–72), the author develops a set of necessary and sufficient conditions that A, B, C must satisfy in order that the curves obeying the above differential equation be identifiable with the totality of extremals connected with a variation problem of second order in the plane. When the conditions are fulfilled, the solution is unique up to an additive arbitrary total derivative with respect to x of a function of x, y, y'. Examples of extremal and nonextremal families of the above quadratic type are given. The paper also contains some necessary conditions for, as well as geometric characterizations of, the ∞^6 and ∞^{2n} curves constituting, respectively, the totality of extremals of variation problems of the third and nth orders in the plane. (Received March 6, 1942.)

176. A. N. Lowan, H. E. Salzer and Abraham Hillman: *Coefficients of differences in the expansion of derivatives in terms of advancing differences.*

The exact values of the coefficients $A_{n,r}$ of the rth differences in Markoff's formula for the nth derivative in terms of advancing differences were computed for $n=1, 2, \cdots, 20$ and $r=n, n+1, n+2, \cdots, n+20$ by the Mathematical Tables Project, Work Projects Administration, New York City. The coefficients are defined by $A_{n,r} = (n/r(r-n)!)\, B^n_{r-n}$ where B^r_{r-n} is the $(r-n)$th Bernoulli number of the rth order. These coefficients are useful in computing the derivatives of functions from tabulated values. In particular, they constitute a powerful tool for calculating values of functions for complex values of the argument when the function is known to be analytic

in the neighborhood of the real axis and when the function has been tabulated for real arguments with sufficient accuracy and to a sufficiently small interval. Coefficients were computed with the aid of the recurrence formula $A_{n,r+1}=(nA_{n-1,r}-rA_{n,r})/(r+1)$ and further checked against the values obtained from the factorial polynomials. The usefulness of the table of coefficients here discussed will be enhanced by a table of $(x+iy)^n$ the preparation of which is now contemplated. (Received February 20, 1942.)

177. G. T. Miller and H. K. Hughes: *Analytic continuation of functions defined by factorial series.*

Given a function of z defined by a factorial series of the form: $\Omega(z)=\sum_{m=0}^{\infty}(-1)^{m(k+1)}g(m)/z\cdot(z+1)\cdots(z+m)$, where $g(w)$ is a single-valued analytic function of the complex variable $w=x+iy$, for $x\geqq -1/2$, and $g(w)$ is such that for all such x and for $|y|$ sufficiently large, $|g(x+iy)/g(x)|<M\exp(\pi(k+(1/2))-\epsilon)|y|$ then an analytic continuation of the function $\Omega(z)$ is given by the sum of two infinite integrals. This result may be considered to be an extension of a result obtained by H. K. Hughes (American Journal of Mathematics, vol. 53 (1931), pp. 771–775). The method of proof is based on the calculus of residues and bears a close analogy to the work of W. B. Ford (*Asymptotic Developments*, University of Michigan Studies, 1936) and to that of C. V. Newsom (American Journal of Mathematics, vol. 60 (1938), pp. 561–572). In case the function $g(w)$ has singularities, certain loop integrals are added to the expression already obtained. (Received March 10, 1942.)

178. Marston Morse and G. M. Ewing: *The non-regular case in the variational theory in large.*

The original treatment both of the absolute minimum theory and of the more general variational theory in the large made use of broken extremals. In the non-regular case these lost their most useful characteristics, making it necessary to treat the variational theory in the large in a new way. This has been done in three papers, the first on functional topology by Morse which makes the necessary alterations in the underlying topological theory, and in two papers by Morse and Ewing dealing with the local analytical aspects of the problem such as upper reducibility and the generalized Euler or homotopy theorem. The classical minimum theory including the Lindeberg theorem is first reviewed and refined, making a more systematic use of the hypothesis of convexity of the integrand than previously. In the second paper the more difficult question of upper-reducibility is taken up. The final theorems are on a general non-regular problem on a Riemannian manifold. (Received March 16, 1942.)

179. Isaac Opatowski: *Confluent hypergeometric functions and Markhoff chains.*

New simple relations between confluent hypergeometric functions are obtained by means of the Laplace transformation. The study is based on the function $[(n+m-1)!]^{-1}t^{n+m-1}e^{-t}\ {}_1F_1(m,\ n+m,\ t)\equiv\gamma_m(t,\ n)$, which has certain advantages of simplicity with respect to the Erdélyi's function $N_{k,m}$ (Mathematische Zeitschrift, vol. 42 (1936), pp. 125–143) for example, $\gamma_m(t,\ n)*\gamma_p(t,\ q)=\gamma_{m+p}(t,\ n+q)$, where $*$ means the convolution. Then γ_1 is the Pearson incomplete gamma function and $\gamma_{m+1}(t,\ n)$ is the m times repeated integral of $\gamma_1(t,\ n)$ between $t=0$ and $t=t$ (the integration being understood for an arbitrary m as the general derivative of Liouville-Riemann). The function $\gamma_m(t,\ n)$ is used to express the probability function $P_i(t)$

and calculate the moments of the Markoff chain (a generalization of the Poisson distribution): $dP_i/dt = k_iP_{i-1} - k_{i+1}P_i$, where $k_i = 0$ and the positive constants k_i's have for $i = 1, 2, 3, \cdots$ two different values only. (Received March 20, 1942.)

180. E. J. Pinney: *Calculus of variations in abstract spaces and related topics.* I.

The calculus of variations problem considered is that of minimizing the integral $I(y) = \int f(x, y(x), y_x(x))dV(x)$, where $y(x)$ is a function on an n-dimensional arithmetic space R to a linear topological space T which is required to contain a convex open set properly, $y_x(x)$ denotes the set of linear topological derivatives, $f(x, h, h_x)$ is a functional on R, T, T, $\cdots$, T, and the integral is an n-dimensional Lebesgue integral. The development necessitates the proof of certain theorems about the integral, and this first paper is devoted to that. The familar expression for the transformation of an integral under a mapping of the set on which it is defined to another (measurable) set is established. Green's theorem is established. A topological differential equivalent to a Fréchet differential when T is a Banach space is studied. The theorem of the mean is established for certain functionals, and continuity under the integral sign is considered. (Received March 14, 1942.)

181. E. J. Pinney: *Calculus of variations in abstract spaces and related topics.* II.

In this continuation of the first paper under this title, the theory there developed is applied to the calculus of variations problem mentioned there. The function $f(x, h, h_x)$ is defined, and an allowable class of functions $y(x)$ is defined. The abstract analogues of the necessary conditions of Euler, Weierstrass, Legendre, and Jacobi are established, and in addition the analogue of the necessary condition of Carathéodory is established. This condition, which reduces to that of Weierstrass in the case of the single integral, doesn't seem to be very well known as yet. Two "corner" conditions are established, and the condition of transversality on the boundaries is established. An investigation of sufficiency conditions is projected for the future. (Received March 14, 1942.)

182. Harry Pollard: *An inversion formula for the Stieltjes transform.*

This treatment of the Stieltjes transform (1) $f(x) = \int_0^\infty (x+t)^{-1}d\alpha(t)$ is intended to parallel a recent study of the Laplace integral made by Boas and Widder. The underlying idea, suggested by the work of Paley and Wiener, is to consider the iterate of the transform and to apply known results to this. In this way a new inversion operator can be derived for the transform (1). In terms of this operator necessary and sufficient conditions are obtained for the representation of a function as a Stieltjes transform with $\alpha(t)$ of general or of preassigned type. All these results depend on a knowledge of $f(x)$ on the real axis. (Received March 3, 1942.)

183. Harry Pollard: *On subseries of a convergent series.*

In a recent paper J. D. Hill (this Bulletin, vol. 48 (1942), p. 103) has defined the mean-value m of all the subseries of a given absolutely convergent series $\sum u_n$. In this case he has shown that $m = s/2$, where s is the sum of the series. By use of well known properties of the Rademacher functions this result is extended to the case of all convergent series for which $\sum |a_n|^2 < \infty$. If $\sum |a_n|^2 = \infty$, m fails to be defined, so that the extension is best possible. (Received March 4, 1942.)

184. George Pólya: *On the combinatory analysis of classifications and permutations.*

Let K_n denote the number of classifications of a set of n objects, $K_n=1, 2, 5, 15, \cdots$ for $n=1, 2, 3, 4, \cdots$ and let $K_0=1$. These K_n and connected combinatorial numbers occur in many questions, for example, K_n is the coefficient of $x^n e/n!$ in the development of $\exp(\exp(x))$. Let G be a permutation group of order g and the degree n, let $\mu_0=1$, and μ_k be the arithmetic mean of the kth powers of the traces of its g permutations. The μ_k are integers, $\mu_1 \geqq K_1$, $\mu_2 \geqq K_2, \cdots, \mu_n \geqq K_n$. If the equality is valid in all n inequalities, G is the symmetric group. Otherwise there is just one p, $0 \leqq p < n$, such that $\mu_p = K_p$, $\mu_{p+1} < K_{p+1}$ and G is p-fold transitive. (Received March 17, 1942.)

185. H. A. Rademacher: *On the Bloch-Landau constant.*

Landau in 1929 in a paper on the Bloch constant $\mathfrak{B}$ introduced another constant $\mathfrak{L}$ through the following definition: There exists a constant $\mathfrak{L}$ such that every function $w=f(z)$ which is regular in $|z|<1$ with $f'(0)=1$ covers completely, for any $\epsilon>0$, a circle of radius $\mathfrak{L}-\epsilon$ in the w-plane, whereas to every $\epsilon>0$ there exists a function of the same specifications whose values do not completely fill any circle of radius $\mathfrak{L}+\epsilon$. Landau gave the upper estimate $\mathfrak{L} \leqq 2^{1/2}3^{-1}\Gamma^3(1/3)\Gamma^{-4}(1/4)=0.55488\cdots$. In this paper we find $\mathfrak{L} \leqq \Gamma(5/6)\Gamma(1/3)\Gamma(1/6)^{-1}=0.54325\cdots$. This upper bound is found by a suitable example obtained through the conformal mapping of a zero-angled circular triangle on a straight equilateral triangle. The calculations can be carried out with the help of a formula given by Ahlfors and Grunsky in 1937. (Received February 27, 1942.)

186. T. Radó: *On a problem of Geöcze.*

The Lebesgue area $A(S)$ of a continuous surface S is defined as the greatest lower bound of $\liminf E(\pi_n)$ for all sequences of polyhedra π_n converging to S in the Fréchet sense, where $E(\pi_n)$ stands for the elementary area of π_n. Denote by $A^*(S)$ the quantity obtained by requiring that the polyhedra π_n be inscribed in S. The problem of determining whether $A^*(S)=A(S)$ was first attacked by Geöcze (Mathematische und Naturwissenschaftlische Berichte aus Ungarn, vol. 26 (1910), pp. 1–88). The purpose of this paper is to extend the results of Geöcze by proving the following theorem. Let S be given by $z=f(x, y)$, where $f(x, y)$ is continuous in the unit square and is absolutely continuous in x for almost every y, or alternatively, absolutely continuous in y for almost every x. Then $A^*(S)=A(S)$. (Received February 28, 1942.)

187. P. V. Reichelderfer: *On bounded variation and absolute continuity for parametric representations of continuous surfaces.*

If S be a continuous surface in $x^1x^2x^3$-space given in parametric representation, denote by ${}^1T$, ${}^2T$, ${}^3T$ the corresponding representations for the projections of S upon the x^2x^3-, x^3x^1-, x^1x^2-planes, respectively. Definitions for essential variation and essential absolute continuity are given for ${}^1T$, ${}^2T$, ${}^3T$, based upon ideas developed by T. Radó and P. Reichelderfer (Transactions of this Society, vol. 49 (1941), pp. 258–307). An essential area $eA(S)$ is defined for S; it is a lower semi-continuous functional of S. A necessary and sufficient condition that $eA(S)$ be finite is that each of ${}^1T$, ${}^2T$, ${}^3T$ be of bounded essential variation. If $eA(S)$ is finite, then the generalized Jacobians

${}^1J$, ${}^2J$, ${}^3J$ for ${}^1T$, ${}^2T$, ${}^3T$ exist almost everywhere and are summable, and $eA(S) \geqq \iint [{}^1J^2+{}^2J^2+{}^3J^2]^{1/2}$. A necessary and sufficient condition that the sign of equality hold here is that each of ${}^1T$, ${}^2T$, ${}^3T$ be essentially absolutely continuous. If S has a non-parametric representation, then $eA(S)$ equals the Lebesgue area of S, and this theory becomes equivalent to that developed by Tonelli for this special case. (Received March 18, 1942.)

188. W. H. Roever: *Comment on the derivation of the law of perfect gases.*

In this paper the author shows that, to derive from the laws of Boyle and Charles the law of perfect gases, one encounters the problem of showing that the two two-parameter families of curves defined by these laws lie on a one-parameter family of surfaces. Also the derivation of the expression for the entropy of a perfect gas is discussed. (Received March 6, 1942.)

189. H. M. Schwartz: *On some general series expansions.* Preliminary report.

Using results obtained in two earlier papers (*Sequences of Stieltjes integrals*, this Bulletin, vol. 47 (1941), pp. 947–955, and abstract 48-1-55), the author studies some questions of convergence and summability of expansions in series of functions which form an orthogonal set with respect to a function of bounded variation, by methods similar to those employed by H. Lebesgue in his paper *Sur les intégrales singulières* (Annales de Toulouse, (3), vol. 1 (1909)). (Received March 20, 1942.)

190. M. F. Smiley: *A comparison of algebraic, metric, and lattice betweenness.*

It is shown that metric betweenness (see L. M. Blumenthal, *Distance Geometry*) and lattice betweenness (see Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, forthcoming in the Transactions of this Society) coincide in a complete normed real vector lattice if and only if this lattice is equivalent to an (L)-space (S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Annals of Mathematics, (2), vol 42 (1941), pp. 523–537). The ranges of coincidence of algebraic and metric and of algebraic and lattice betweenness are also determined. Metric betweenness in the Banach space $C[0, 1]$ is investigated. (Received March 12, 1942.)

191. M. F. Smiley: *A remark on S. Kakutani's characterization of (L)-spaces.*

The important condition IX of Kakutani (*Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Annals of Mathematics, (2), vol. 42 (1941), pp. 523–537) may be replaced by the equivalent condition (1) $\|x-y\| = \|x \vee y - x \wedge y\|$. It is shown that this permits an economy of assumptions in §§3–5 of the paper cited. The proofs are based on the fact that the condition (1) is closely related to a fundamental property of modular functionals (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, 1940, p. 40). (Received March 12, 1942.)

192. W. S. Snyder: *Non-parametric surfaces of bounded variation.*

The paper shows that if $f(x, y)$ is of bounded variation in the sense of Hardy and Krause (Hobson, *Functions of a Real Variable*, vol. 1, 3rd edition, 1927, §254), then the Lebesgue area of the surface $z=f(x, y)$ is equal to the Burkill integral of a certain rectangle function analogous to the rectangle function of de Geöcze (Saks, *Theory of the Integral*, p. 171). The method shows that the area of the surface may be calculated by a single passage to the limit from the areas of certain polyhedra inscribed in the surface. (Received March 20, 1942.)

193. Otto Szász: *On sequences of polynomials and the distribution of their zeros.*

The author considers sequences of polynomials $P_n(z)=c_{n0}+c_{n1}z+c_{n2}z^2+\cdots$ with increasing degrees $m_1<m_2<\cdots$, respectively, where the roots of P_n lie in a half-plane containing the origin on its boundary line, the direction of which may depend on n. If the first three coefficients are uniformly bounded, and the c_{n0} are bounded away from 0, then the sequence is uniformly bounded in any finite domain. If, in addition $\lim c_{n\nu}$ exists as $n\to\infty$ for each ν, then the sequence converges uniformly in any finite domain, thus representing an entire function, the convergence exponent and order of which are at most two. In particular if the sequence is a subsequence of partial sums of a formal power series, then this series represents an entire function of the described type. Some previous results of G. Pólya are thus generalized. (Received March 6, 1942.)

194. S. M. Ulam and D. H. Hyers: *Approximate isometries of the space of continuous functions.*

A transformation of a metric space E into itself will be called an ϵ-isometry if it changes distances by an amount less than ϵ (see abstract 47-9-427). Properties of such transformations are studied in the case where E is the space of continuous functions. (Received March 24, 1942.)

195. F. A. Valentine: *On the extension of a vector function so as to preserve a Lipschitz condition.*

Let $f(x)$ be a vector function mapping a set S in the euclidean plane into a set S' in the euclidean plane. Moreover $f(x)$ satisfies a Lipschitz condition of the form $|f(x_1)-f(x_2)|\leqq K|x_1-x_2|$ for all pairs x_1 and x_2 in S, where $|x-y|$ is the euclidean distance between the points x and y. If T is any set containing S, then $f(x)$ can be extended to T so as to preserve the Lipschitz condition. The extension $[f(x), x\in T]$ can be defined so as to be contained in any prescribed bounded, closed, convex set containing $S'\equiv[f(x), x\in S]$. This extension is a consequence of the following: *Consider a set of circles in the plane, such that there is a point in common to all of the circles. Move these circles to new positions in the plane, subject to the condition that the distance between any pair of centers is not increased. Then all of the circles in the new positions will still have a point in common.* After proving this result for three circles, one can prove it for an arbitrary set by using a theorem of Helly for families of bounded, closed, convex sets. (Received March 6, 1942.)

196. J. L. Walsh: *On the overconvergence, degree of convergence, and zeros of sequences of analytic functions.*

The entire theory of overconvergence as developed by Porter, Jentzsch, Os-

trowski, Bourion, and others is here generalized and unified by the concept of exact harmonic majorant of a sequence of analytic functions. If the function $V(z)$ is harmonic in a region R of the z-plane, if the functions $F_n(z)$ are locally single-valued and analytic in R except for branch points, and if $|F_n(z)|$ is single-valued in R, then $V(z)$ is said to be an exact harmonic majorant of the sequence $F_n(z)$ in R provided one has $\limsup_{n\to\infty}[\max |F_n(z)|, z \text{ on } Q] = [\max e^{V(z)}, z \text{ on } Q]$ for every continuum Q (not a single point) in R. Applications of this concept involve degree of convergence and properties of the zeros of functions, and include maximal sequences of polynomials and of other rational functions, and many other sequences of analytic functions. (Received March 16, 1942.)

197. M. S. Webster: *A convergence theorem for certain Lagrange interpolation polynomials.*

A convergence theorem for a sequence of Lagrange interpolation polynomials based on the zeros of a sequence of certain Jacobi polynomials is proved. The method and result are similar to a theorem of Grünwald (this Bulletin, vol. 47, (1941), pp. 271–275). (Received March 19, 1942.)

198. Hermann Weyl: *Solution of the simplest boundary-layer problems in hydrodynamics.*

For some simple configurations the hydrodynamic boundary-layer problem can be reduced to a non-linear ordinary differential equation of third order involving a parameter λ. For $\lambda=0$ and $1/2$, solution may be obtained by a rapidly converging process of alternating successive approximations. The general case is attacked by a suitable adaptation of the method of fixed points of transformations in functional spaces. (Received February 28, 1942.)

199. František Wolf: *On the limits of harmonic and analytic functions along radii which form a set of positive measure.*

If $u(r,\theta)=\log|f(re^{i\theta})|$ and $f(z)$ is analytic in the unit circle $r<1$, $u(r,\theta)\leqq M/(1-r)^n$ for any M and n, and $\limsup_{r\to 1} u(r,\theta)\leqq 0$ for $\theta\subset E$, $|E|>0$, then $\limsup u(r,\theta)\leqq 0$ in any sector at almost all points of E. Hence if $u(r,\theta)$ is harmonic and satisfies the conditions of the theorem, then $u(r,\theta)$ and its conjugate $v(r,\theta)$ have finite limits in any sector at almost all points of E. This follows from above by the well known results of Privaloff (Recueil Mathématique de Moscou, vol. 91 (1923), p. 232) and Fatou. Another corollary is: If $f(z)$ is analytic in $|z|<1$, $|f(z)|\leqq\exp[M/(1-r)^n]$, and $\lim_{r\to 1} f(re^{i\theta})=0$ for $\theta\subset E$, $|E|>0$, then $f(z)\equiv 0$. (Received March 20, 1942.)

Applied Mathematics

200. Stefan Bergman: *Determination of pressure in the two-dimensional flow of an incompressible perfect fluid.*

The author considers a flow of an incompressible perfect fluid around a wing profile. The pressure distribution is determined by the function $W(z)$ which maps the exterior $\mathfrak{E}$ of the wing profile onto the exterior $\mathfrak{R}$ of a circle. Of particular interest is the evaluation of $W(z)$ on the boundary in the neighborhood of the vertex O. Let boundary in the neighborhood of O be formed by two circular arcs CO and BO which make an angle α at O. Let O and D be the intersections of the circles on which the arcs CO and BO lie. Suppose that arcs CD and BD lie inside of the profile (Hypothesis

H). Then a linear transformation followed by one of the form $\zeta^* = \zeta^{\pi/(2\pi-\alpha)}$ will map $\mathfrak{E}$ into a domain $\mathfrak{T}$ which lies in the upper half-plane, and the arcs OA and OC will become segments O_2A_2 and O_2C_2 of the real axis. Using orthogonal functions (see Bergman, Publication of Brown University, 1941, p. 118), the author determines the function $w(\zeta^*)$ mapping $\mathfrak{T}+\mathfrak{T}^*$ into the unit circle. $\mathfrak{T}^*$ is the domain obtained from $\mathfrak{T}$ by reflection on the real axis. Then $(1/2)[w(\zeta^*)+(w(\zeta^*))^{-1}]^{-1}$ maps $\mathfrak{T}$ into $\mathfrak{R}$. If H is not fulfilled, follow the linear transformation by one which maps the triangle $O_1A_1C_1$ of the ζ-plane into the upper half-plane. (Received February 21, 1942.)

201. Stefan Bergman: *Three-dimensional flow of a perfect incompressible fluid and its singularities.*

The author considers vectors $\mathfrak{S}(\mathfrak{X})$, $\mathfrak{X}=(x_1, x_2, x_3)$. The components of $\mathfrak{S}$ are harmonic functions with an algebraic singularity described in the author's papers (Mathematische Zeitschrift, vol. 24 (1925–1926), p. 655 and Mathematische Annalen, vol. 99 (1928), p. 645). At every point outside of the singularity, curl $\mathfrak{S}=0$, div $\mathfrak{S}=0$ holds. Let $\theta(\mathfrak{X})$ be a potential function such that grad $\theta(\mathfrak{X})=\mathfrak{S}(\mathfrak{X})$. The author defines a curve $l^1(\zeta)$ in the $x_1x_2x_3$-space for every $\mathfrak{S}(\mathfrak{X})$ and every complex number ζ. Let q^1 be a closed curve in the ζ-plane. $L^2(q^1)$ is the logical sum of $l^1(\zeta)$, $\zeta \neq q^1$, so that to every point $\mathfrak{X}$ of $L^2(q^1)$ corresponds a $\zeta=\zeta(\mathfrak{X})$. [The singularity line of $\mathfrak{S}(\mathfrak{X})$ lies on $L^2(q^1)$.] Finally the author defines for $\mathfrak{S}(\mathfrak{X})$ the residue functions $R_\nu(\zeta)$. Let i^1 be a closed curve in the schlicht $x_1x_2x_3$-space. i^1 is open in the multiply-covered space M^3 in which $\mathfrak{S}(\mathfrak{X})$ is univalent. Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be both end points of i^1 in M^3. Then $\int_{i^1}\mathfrak{S}(\mathfrak{X})\cdot d\mathfrak{X}+\sum[\theta_k(\mathfrak{E}_2)-\theta_k(\mathfrak{X}_k^*)]+\sum[\theta_n(\mathfrak{X}_n^*)-\theta_n(\mathfrak{E}_1)]=\sum\int_{\zeta(\mathfrak{X}_\mu^*)}^{\zeta(\mathfrak{X}_{\mu+1}^*)}R_\mu(\zeta)d\zeta$. Here $\theta_m(\mathfrak{X})$ are functions connected with $\theta(\mathfrak{X})$, and $\mathfrak{X}_m^*$ intersections of i^1 with $L^2(q^1)$. An analogous relation holds for a vector $\mathfrak{H}$, $\mathfrak{H}=\mathfrak{H}_1+\sum_{\nu=1}^{N}\mathfrak{S}_\nu$, where $\mathfrak{H}_1$ is a regular vector and $\mathfrak{S}_\nu$ the above described vectors. (Received March 6, 1942.)

202. Hilda P. Geiringer: *On the numerical solution of linear problems by group iteration.*

The so called Ph. Seidel iteration method for solving systems of linear equations converges towards their solution for all systems originated from a minimum problem; for example, in the following groups of problems: (1) statistical (least square) problems, (2) problems of mechanics, for example, statically determined or indeterminate frameworks, (3) elliptic boundary value problems. For the second group the method reduces the solution of an n-fold indeterminate system to the successive solution of simply indeterminate systems. (a) Each of these simply indeterminate systems can be solved in whatever way seems appropriate, not necessarily through the respective Maxwell equation. (b) Introduce "group-iteration" (see H. Geiringer, *Zur Praxis der Lösung linearer Gleichungen in der Statik*, Zeitschrift für Angewandte Mathematik und Mechanik, 1928, and R. von Mises and H. Geiringer, *Praktische Verfahren der Gleichungsauflösung*, ibid., 1929); that is, instead of reducing to simply indeterminate problems, use as intermediate steps r-fold indeterminate systems whose solutions may be found by any suitable combination of equations. Thus liberty of arrangement results and the convergence is accelerated.—It seems that this procedure contains the essentials of R. V. Southwell's "relaxation method" successfully applied by him since 1933 to many mechanical problems and recently presented in a comprehensive work. The group iteration method combined with the use of computing machines to solve the small groups of equations leads to very good results. (Received February 9, 1942.)

203. A. M. Gelbart: *Bounds for pressure in a two-dimensional flow of an incompressible perfect fluid.*

It is known that the problem of the pressure distribution along a wing in the case of a two-dimensional flow of an incompressible perfect fluid can be reduced to the problem of determining the function which maps the exterior domain into the exterior of a circle. This paper deals with some properties of the function in the neighborhood of the angle of the wing. Bergman treats this problem by employing orthogonal functions and certain special transformations. (See abstract 48-5-200.) Using this approach, some inequalities previously obtained by the author for the coefficients of the mapping function, and a minimum integral, inequalities for the velocity in the neighborhood of the angle are obtained which depend only upon a suitable domain in which the boundary of the profile lies. (Received March 7, 1942.)

204. W. A. Mersman: *Heat conduction in a finite composite solid.*

The problem of one-dimensional heat conduction in a composite wall has been solved by Churchill (Duke Mathematical Journal, vol. 2 (1936), pp. 405–414, and Mathematische Annalen, vol. 115 (1938), pp. 720–739), the solution being presented in the form of a series which converges rapidly for large time values. The present paper furnishes a transformation of Churchill's solution in the form of a series which converges rapidly for small time values. This is done by expanding the Laplace transform of the solution as a geometric series and inverting term-by-term, instead of applying the Mittag-Leffler theorem and the inversion theorems of Doetsch and Churchill. (Received February 19, 1942.)

205. W. A. Mersman: *Heat conduction in an infinite composite solid with an interface resistance.*

The problem of one-dimensional heat conduction in a doubly infinite composite solid with an interface resistance is solved by the Laplace transformation method. The interface conditions are: (1) the product of conductivity and temperature gradient is continuous across the interface; (2) the temperature discontinuity across the interface is proportional to the product in (1) above, the factor of proportionality being a constant. (Received February 9, 1942.)

GEOMETRY

206. Herbert Busemann: *Spaces with convex spheres.*

In a metric space a continuous curve which is locally isometric with a euclidean straight line will be called a geodesic. The paper considers a finitely compact metric space in which there is exactly one geodesic through any two different points. With an obvious definition of a tangent of a sphere a sphere is called convex if no tangent of the sphere contains interior points of that sphere. Assume that all spheres are convex and that the space has dimension greater than or equal to 3. The space is congruent to an elliptic space, as soon as at least one geodesic is closed. If all geodesics are open, convexity of the spheres as defined above coincides with the usual idea that a segment whose end points are in a sphere lies completely in the sphere, but the space is not necessarily flat. However if the parallel axiom (properly formulated) holds, the space is flat and its metric is Minkowskian. (Received March 20, 1942.)

207. Aaron Fialkow: *Conformal differential geometry of a subspace.*

In previous papers and abstracts, the author has shown that a conformal differentiation process may be defined with respect to an arbitrary subspace V_n in any Riemann space V_m $(0<n<m;\ m>2)$. By means of this tool, the conformal fundamental forms of V_n, which completely characterize its conformal geometry, were discovered as well as the fundamental differential equations which the coefficients of these forms satisfy. The conformal differential geometry of V_n based upon these forms and upon the conformal differentiation process is now developed. As in classical differential geometry, it includes the conformal analogues of normal and geodesic curvature, conjugate and asymptotic directions, developables and spaces of constant curvature, and so on. Classical theorems such as those of Meusnier, Euler and others hold in this geometry. However, some completely new geometric theorems are derived which are based on tensors having no analogue in the classical theory. Many results of a special character (particularly if $n=2$) are obtained. Some important special conformal coordinate systems are indicated. (Received March 5, 1942.)

208. L. C. Hutchinson: *On the linear line complex in n-space.*

This paper starts from certain contradictions in the literature and from the careful distinction between two closely related, but not identical concepts, a bivector and its associated $(n-2)$-complex, and develops the general theory employing tensor methods. (Received March 7, 1942.)

209. A. D. Michal: *A theory of fluctuations in Riemannian spaces.* Preliminary report.

This paper deals with parallel displacement and differential invariants of multiple point correlation tensor fields in n-dimensional Riemannian geometry. (Received March 14, 1942.)

210. Don Mittleman: *Spin in newtonian mechanics.*

Newtonian mechanics has not considered the possibility of attributing spin properties to point particles moving in a general positional field of force. However, this may be done and the appropriate equations are obtained by considering the limiting form of the equations of motion of a rigid body of finite dimensions as the maximum diameter of the body approaches zero. The result is dependent on the limit of the ratio of the principal moments of inertia, but to fix the limit it is assumed that the ratio of the principal axes of the ellipsoid of inertia approaches one. For a particle, the trajectory is no longer a curve but a series of elements. The geometry of the trajectorial series is investigated. The program is repeated for infinitesimal rods. (Received February 2, 1942.)

211. T. W. Tomlinson: *Geometry of linear fractional polygenic functions.*

Linear fractional functions of $z=x+iy$ and $\bar{z}=x-iy$ are studied in detail. Any such correspondence T can be factored into a Moebius transformation followed by a projectivity. In general T possesses one point f_1 of direct conformality and one point f_2 of reverse conformality. The Kasner circles corresponding to the points of the perpendicular bisector of f_1f_2 pass through the origin and their centers are on a circle through the origin. (Received March 20, 1942.)

212. C. B. Tompkins: *Local imbedding of Riemannian spaces.*

The paper gives a proof of Janet's theorem on local imbedding of an n-dimensional Riemannian space in euclidean space of $n(n+1)/2$ dimensions. The proof is by induction on the number of dimensions, and the induction is made possible by strengthening the theorem slightly to state that the Riemannian space can be imbedded so that its tangent plane and the vectors obtained by differentiating the imbedding functions twice with respect to a set of $(n-1)$ of the n parameters of the Riemannian space together span the euclidean space. The proof involves an existence theorem in differential equations, a warping process somewhat analogous to the process of rolling a plane into a cylinder, and an algebraic lemma which is used to show that the warping process will furnish the independence of vectors required in the strengthened theorem of the inductive proof. (Received March 20, 1942.)

213. B. J. Topel and P. M. Pepper: *Imbedding theorems under weakened hypotheses.*

The congruent imbedding of a semi-metric space into E_n, the euclidean n-space, as implied by the imbeddability of as few as possible of its $(n+2)$-tuples is considered. For S, consisting of $n+4$ or more points, to be imbeddable into E_n, but not E_{n-1}, it is necessary and sufficient that S contain a certain nucleus S' and that all $(n+2)$-tuples containing at least n points of S' be imbeddable. Sufficient conditions are derived for the imbeddability of S when an upper bound is placed on the number of non-mapping $(n+2)$-tuples with no restriction on their distribution in S. These theorems sharpen Menger's quasi-congruence theorem. Necessary and sufficient conditions are determined for the imbeddability of a semi-metric space into E_n. Similar theorems are considered for imbeddability into an element of any congruence system. The structure of non-mapping sets with the minimum number of non-mapping $(n+2)$-tuples is studied. (Received March 23, 1942.)

214. J. E. Wilkins: *A characterization of the quadric of Wilczynski.*

The quadric of Wilczynski at a point P of a non-ruled surface S in projective three-dimensional space is characterized as the unique quadric having second-order contact with S at P (therefore intersecting S in a curve with a triple point at P), the three triple-point tangents being the tangents of Darboux, and the three triple-point osculating planes having a line in common. This line is found to be the canonical line of the first kind with $k=5/12$. (Received March 12, 1942.)

Logic and Foundations

215. S. C. Kleene: *On the forms of the predicates in the theory of constructive ordinals.* Preliminary report.

In the system S_3 of notation for ordinal numbers (Journal of Symbolic Logic, vol. 3 (1938), p. 155), the class O of the natural numbers which represent ordinals, and the partial ordering relation $<_o$ between such numbers, were defined by a transfinite induction. It is now shown that the predicates $a \in O$ and $a <_o b$ are expressible explicitly in the respective forms $(x)(Ey)R(a, x, y)$ and $(x)(Ey)S(a, b, x, y)$ where R and S are primitive recursive predicates. The result can be used to exhibit the incompleteness of ordinal logics by a method presented previously. (See abstract 46-11-464. Erratum: for "for." read "for all.") The proof illustrates a technique to

which recourse may be had generally in attempts to reduce inductive definitions to explicit definitions in terms of recursive predicates and quantifiers. (Received March 3, 1942.)

STATISTICS AND PROBABILITY

216. J. H. Bushey: *The distribution function of the mean under the type α hypothesis.*

An orthogonal expansion (type α series) with the Pearson type III function as weight function is obtained in a form suitable to utilize the tables of Salvosa in representing population frequencies. The expansion differs in certain respects from that of Romanovsky. The distribution function of the mean is obtained for samples of n drawn at random from a population represented by the type α series. In special cases this distribution function reduces to that obtained for the Charlier type A series by Baker and to that obtained by Church for the type III function. (Received March 6, 1942.)

217. J. H. Bushey: *The distribution function of the sample total under the type β hypothesis.*

The orthogonal polynomials $\phi_n(x)$ are defined by the weight function $p(x)=C_{s,x}p^x(1-p)^{s-x}$, $(x=0, 1, 2, \cdots, s)$ and the orthogonal relation $\sum_{x=0}^{s}p(x)\cdot\phi_m(x)\phi_n(x)=0$, $m\neq n$, or $=1$, $m=n$. The orthogonal expansion (type β series) $f(x)=\sum_{i=0}^{k}A_ip(x)\phi_i'x)$ may be used as a statistical hypothesis. The Charlier type A series is a special case of both the type β and the type α series (the type α series is reported in another abstract). Another special case of the type β series is the Charlier type B series with the Poisson weight function $p(x)=(e^{-sp}(sp)^x)/x!$. The distribution function for the total $z=n\bar{x}$ for samples of n drawn at random from a population represented by a type β series is obtained. This result includes, as special cases, the distribution function of z for the Charlier type B series and that obtained by Baker for the Charlier type A series. (Received March 6, 1942.)

218. J. H. Bushey: *The products of certain discrete and continuous orthogonal polynomials.*

The discrete orthogonal polynomials $\phi_n(x)$, $(x=0, 1, 2, \cdots, s)$, are defined by the weight function $p(x)=C_{s,x}\cdot p^x(1-p)^{s-x}$ and the orthogonal relation $\sum_{x=0}^{s}p(x)\cdot\phi_m(x)\phi_n(x)=0$, $m\neq n$, or $=1$, $m=n$. The polynomials $\phi_n(x)$ are closely related to the continuous polynomials of Jacobi, Hermite, and Laguerre and have applications in statistics. The product $\phi_m(x)\cdot\phi_n(x)$ is developed in terms of the polynomials $\phi_i(x)$ for $p=q=1/2$ (symmetric polynomials). This development permits the evaluation of the sum $\sum_{x=0}^{s}p(x)\phi_m(x)\phi_n(x)\phi_r(x)$. The corresponding results of Feldheim for Hermite polynomials follow as special cases. (Received March 6, 1942.)

TOPOLOGY

219. E. G. Begle (National Research Fellow): *Intersections of absolute retracts.*

Aronszajn and Borsuk have shown (Fundamenta Mathematicae, vol. 18 (1932) pp. 193–197) that if A and B are compact metric spaces such that their intersection, $A\cap B$, is an absolute retract, then their sum, $A\cup B$, is an absolute retract if and only

if A and B are. They left open the following question: Is $A \cap B$ an absolute retract if A, B and $A \cup B$ are? In this paper an example is presented which answers this question in the negative. On the other hand, it is shown that if A, B and $A \cup B$ are absolute homology retracts, then so is $A \cup B$. (Received March 19, 1942.)

220. M. M. Day: *Compactness of a space with monotone closure.*

A closure c in a set X associates with each $X' \subset X$ a set $cX' \subset X$; c is monotone if $X' \subset X''$ implies $cX' \subset cX''$; c is additive if $c(X'+X'')=cX'+cX''$ for every X', X''. If c is monotone there is a unique largest additive c' such that $c'X' \subset cX'$ for every X'. A set X is called compact under c (J. W. Tukey, *Convergence and Uniformity in Topology*), if given a family of subsets of X such that finite intersections of sets of the family are non-empty, there is a point common to the closures of all sets of the family. The following theorem holds: X is compact under a monotone c if and only if it is compact under c'. The proof falls into two parts: The conditions given by Tukey for compactness of a space with additive closure still hold for monotone c; in particular, X is compact under c if and only if every ultraphalanx in X converges (in the topology of c) to a point of X. The theorem quoted can then be proved by showing that any ultraphalanx has the same limit points in the topologies of c and c'. (Received April 4, 1942.)

221. R. C. James: *Normability of topological abelian groups.*

In this paper the properties of convexity, boundedness, and normability of topological abelian groups are considered. The existence of a bounded, convex neighborhood of the identity is found to be a necessary condition for normability, and a sufficient condition if taken with connectedness or one of several generating properties. The representation of convex and normable topological abelian groups as subgroups of linear topological spaces is investigated. (Received March 14, 1942.)

222. Samuel Kaplan: *Homologies in metric separable spaces.*

It is shown that the denumerable star-finite coverings form a complete family. By developing a method originating with Čech, homology properties of arbitrary subsets can be obtained by use of "neighborhood coverings." A new type of bounding called Čech bounding is defined for Vietoris cycles as follows: To each Vietoris cycle of the space there corresponds a Čech cycle, determined up to homology; the Vietoris cycle is said to Čech bound if its corresponding Čech cycle bounds. By way of application, the following duality theorem is proved (in the case of closed sets reducing to the Alexander duality): If A is an arbitrary subset of the n-sphere S^n, and there are m k-dimensional Vietoris cycles independent in A relative to Čech homologies, then in $S^n - A$ there are m $(n-k-1)$-dimensional Vietoris cycles independent relative to Čech homologies. Among other results are dualities established between the Čech homology properties of arbitrary subsets of S^n and the Vietoris homology properties of their complements, as well as miscellaneous results as: If a Vietoris cycle z^k fails to Čech bound in a metric separable space, there exists a homeomorphic remetrization in which z^k fails to ϵ-bound for some ϵ. (Received March 19, 1942.)

223. A. N. Milgram: *Dimension of the points with at most n rational coordinates.*

In this paper it is shown that the points in Hilbert space at most n of whose coordinates are rational is an $(n+1)$-dimensional set. This is an extension of the theorem

that the points in Hilbert space all of whose coordinates are irrational form a one-dimensional set, which follows directly from the proof by Erdös for the set with all rational coordinates. (Received March 26, 1942.)

224. J. H. Roberts and Paul Civin: *Sections of continuous collections.*

The following theorem is proved: If G is a continuous collection of closed and compact sets filling a separable metric space X, and if G, regarded as a decomposition space, has dimension n, then there exists in X a closed set C such that for any g in G the set $g \cdot C$ is nonvacuous and consists of at most $n+1$ points. (Received March 20, 1942.)

225. R. L. Swain: *Proper and reductive transformations.*

A (single-valued) transformation T of a point set M is called: *proper* if for each $K \subset M$, $T(K)$ is closed and compact whenever K is; *strong proper* if, in addition, T preserves connectedness; *complete proper* (CP) if an additional condition concerning contiguous points is satisfied. A strong proper transformation of a compact continuous curve is continuous. In spaces which may contain contiguous points (and which satisfy R. L. Moore's axiom set Σ_c), for M to be a compact continuous curve it is necessary and sufficient that M should be the image of an ordinary arc under some CP transformation; also a CP transformation of an arc into itself either leaves a point fixed or interchanges two contiguous points. A transformation T of M is called *reductive* provided that if $X \in T(M)$, and $V \subset M$ is such that X is a limit point of $T(V)$, then $T^{-1}(X)$ contains a limit point of V. A reductive transformation is continuous on the nucleus of a compact closed point set. In a compact space the class of reductive transformations equals the class of proper transformations. Using reductive transformations, one may associate with each point set M an abstract measure called its topological power. (Received March 17, 1942.)

226. G. T. Whyburn: *Coherent and saturated collections.*

A theory of coherence and saturation relative to a given collection G of subsets of a connected separable metric space M is developed. Conditions are found under which the collection G will generate a unique upper semi-continuous decomposition of M and under which the associated transformation will be monotone or non-alternating. These are applied to find necessary and sufficient conditions for the mappability of M onto an arc or a circle by a monotone transformation. As a further application, an existence theorem for saturated collections of continua in a locally connected continuum L is proved which yields in a new way the known (Moore-Roberts) monotone mappability of L into a regular curve. (Received March 4, 1942.)

227. G. T. Whyburn: *On the interiority of mappings.*

A previous theorem of the author (as yet unpublished) is extended to yield the following theorem: Let $f(M) = K$ be continuous where M is locally connected separable and metric and let D be the closure of the set of all points of K of Menger-Urysohn order greater than 2. There exists a countable subset C of $K - D$ such that f is interior at every point of $M - f^{-1}(D + C)$. (Received March 4, 1942.)

228. G. T. Whyburn: *Unitary subcontinua.*

A subcontinuum N of a compact metric continuum M will be called unitary proved that if K is any subcontinuum of M intersecting N we have either $K \supset N$ or

$K \subset N$. For example, M itself (and also any single point of M) is unitary. A study of such subcontinua is made and it is shown, for example that: (1) the property of being a unitary subcontinuum containing a given subset X of M is inducible; (2) for any two given unitary subcontinua A and B there exists a unique least unitary subcontinuum $A+B$ containing both A and B; (3) no nondegenerate proper unitary subcontinuum can have a cut point; (4) the property of being a unitary subcontinuum of M is a monotone and also an interior invariant; (5) if M is irreducible between two points, any monotone interior mapping of M into the interval maps unitary subcontinua into single points of the interval. (Received March 4, 1942.)

229. J. W. T. Youngs: *On parametric representations of surfaces.* II.

This paper is a continuation of an earlier study (abstract 46-5-352). A continuous transformation $R(A)=B$ from the 2-sphere A to a set B in 3-space is called a representation. By the Eilenberg-Whyburn factor theorem, $R(A)=L(M(A))$, where $M(A)=\Sigma$ is monotone and $L(\Sigma)=B$ is light. Two representations $R_1(A)$ and $R_2(A)$ are K-equivalent (Kerekjarto, Acta Szeged, vol. 3 (1927), pp. 49–67) if there is a homeomorphism $H(\Sigma_1)=\Sigma_2$ such that $L_1(\Sigma_1)=L_2(H(\Sigma_1))$. They are F-equivalent if the Fréchet distance between them is zero. The prime end theory is used to define what is meant by the statement that R_1 and R_2 are consistent. The concept has to do with orientation. Suppose that R_1 and R_2 are themselves monotone. Then, if they are F-equivalent, they are K-equivalent and consistent. Moreover the converse is true without the restriction that R_1 and R_2 be monotone. The converse is important from the point of view of applications to the theory of area. (Received March 18, 1942.)

NEW PUBLICATIONS

ADAMS, L. J. See RIETZ, H. L.

AGASHE, D. S. See DIWAN, G. S.

CARSLAW, H. S., and JAEGER, J. C. Operational Methods in Applied Mathematics. Oxford, Clarendon Press; London, Oxford University Press, 1941. 16+264 pp. 17s. 6d.

CRATHORNE, A. R. See RIETZ, H. L.

CURTISS, D. R., and MOULTON, E. J. Essentials of Trigonometry, with Applications. Boston, Heath, 1941. 278 pp. $2.25.

DIWAN, G. S., and AGASHE, D. S. Differential Equations. Bombay, D. S. Agashe, St. Xavier's College, 1941. 10+316 pp.

GRIFFITH, B. A. See SYNGE, J. L.

HAAG, V. H. See THOMAS, J. M.

HAGOOD, M. J. Statistics for Sociologists. New York, Reynal and Hitchcock, 1941. 8+934 pp. $4.00.

JAEGER, J. C. See CARSLAW, H. S.

LANGE, O., McINTYRE, F., and YNTEMA, T. O., edited by. Studies in Mathematical Economics and Econometrics. In Memory of Henry Schultz. University of Chicago Press, 1942. 292 pp. $2.50.

McINTYRE, F. See LANGE, O.

MORGENSTERN, O. See VON NEUMANN, J.

MOULTON, E. J. See CURTISS, D. R.

VON NEUMANN, J., and MORGENSTERN, O. Mathematical Theory of Economic Behavior. Princeton University Press, 1942. 308 pp. $3.75.

RAGHAVACHARI, T. K. An Introduction to Analytical Geometry and Calculus. Madras, Oxford University Press, 1941. 8+192 pp. Rs. 2.

RIETZ, H. L., CRATHORNE, A. R., and ADAMS, L. J. Intermediate Algebra. New York, Holt, 1942. 8+248 pp. $1.75.

SYNGE, J. L., and GRIFFITH, B. A. Principles of Mechanics. New York and London, McGraw-Hill, 1942. 12+514 pp. $4.50.

THOMAS, J. M. Elementary Mathematics in Artillery Fire. (With Tables Prepared by V. H. Haag.) New York and London, McGraw-Hill, 1942. 11+256 pp. $2.50.

YNTEMA, T. O. See LANGE, O.